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HISTORY

Rybnikov, K. A. On the so-called creative and critical periods in the history of mathematical analysis. Istor.-Mat. Issled. 7, 643–665 (1954). (Russian)

Korcik, Antoni. A contribution to the history of propositional calculus. Studia Logica 1 (1953), 247–253 (1954). (Polish. Russian and English summaries)

Morduhaĭ-Boltovskoĭ, D. D. From the past of analytic geometry. Trudy Inst. Istorii Estest. 4, 216–235 (1952). (Russian)

David, F. N. Studies in the history of probability and statistics. I. Dicing and gaming (a note on the history of probability). Biometrika 42, 1–15 (1955).

Galli, Mario. Il valore delle speculazioni galileiane relative alla forza centrifuga. Boll. Un. Mat. Ital. (3) 10, 77–96 (1955).

***Przypkowski, Tadeusz. Dzieje myśli Kopernikowskiej.** [History of Copernican thought.] Wydawnictwo Ministerstwa Obrony Narodowej, Warszawa, 1954. 115 pp. zł. 3.

***Beaujouan, Guy. L'enseignement de l'arithmétique élémentaire à l'Université de Paris aux XIII^e et XIV^e siècles.** Homenaje a Millás-Vallicrosa, Vol. I, pp. 93–124. Consejo Superior de Investigaciones Científicas, Barcelona, 1954.

***Der Briefwechsel von Johann Bernoulli. Bd. I.** Herausgegeben von der Naturforschenden Gesellschaft in Basel. Birkhäuser Verlag Basel, 1955. 531 pp. (6 plates). DM 60.00.

This volume contains an introduction (pp. 9–81) by O. Spiess and Johann Bernoulli's correspondence with Jakob Bernoulli, the Marquis de l'Hôpital, and other miscellaneous correspondence. The letters are copiously annotated and indexed.

***Bianchi, Luigi. Opere. Vol. III. Sistemi tripli ortogonali.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1955. 851 pp. 7500 Lire.

For volumes I and II see MR 14, 524; 15, 276, 591. The volume contains 29 papers on the subject of the subtitle. They are arranged in five groups, each with an introduction by the late Bertrand Gambier.

***Čaplygin, S. A. Izbrannye trudy po mehanike i matematike.** [Selected works on mechanics and mathematics.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 568 pp. (4 plates). 14.60 rubles.

The volume contains eighteen selected papers by Čaplygin, a brief biography by M. V. Keldyš, a list of Čaplygin's works, and some remarks on the papers.

***Čebyšev, P. L. Izbrannye trudy.** [Selected works.] Izdat. Akad. Nauk SSSR, Moscow, 1955. 926 pp. (1 plate). 37.70 rubles.

This volume contains four selected papers on number theory, four on probability theory, ten on various questions of analysis, and eight on mechanisms. In addition there is an essay by N. I. Ahiezer, Čebyšev and his scientific legacy, an article by Artobolevskiĭ and Levitskiĭ, Models of Čebyšev's mechanisms, and a commentary on the papers on mechanisms.

de Broglie, Louis. Notice nécrologique sur Albert Einstein. C. R. Acad. Sci. Paris 240, 1741–1745 (1955).

Bašmakova, I. G., and Yuškevič, A. P. Leonhard Euler. Istor.-Mat. Issled. 7, 453–512 (1954). (Russian)

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Yuškevič, A. P. The last letter of L. Euler to H. Goldbach. Istor.-Mat. Issled. 7, 625–629 (1954). (Russian)

Kostryukov, K. I. On an attempt to publish the works of Leonhard Euler. Istor.-Mat. Issled. 7, 630–640 (1954). (Russian)

Vincensini, Paul. Bertrand Gambier, 1879–1954. Enseignement Math. 40 (1951–1954), 57–61 (1955).

Emersleben, O. Carl Friedrich Gauss (1777–1855) zum 100. Todestag. Wiss. Ann. 4, 121–123 (1955).

Levasseur, Karl. Carl Friedrich Gauss' grundlegende Bedeutung für die Geodäsie. Österreich. Z. Vermessgswes. 43, 1–16 (1 plate) (1955).

In memory of Vladimir Vasil'evič Golubev. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1954, no. 12, 3–18 (1 plate) (1955). (Russian)

Rather detailed discussion of Golubev's scientific contributions and a list of his published works.

Obituary: Vladimir Vasil'evič Golubev. Prikl. Mat. Meh. 19, 129–142 (1 plate) (1955). (Russian)

Brief biography, survey of his scientific work and a list of his scientific papers.

Polubarinova-Kočina, P. Ya. On the biography of S. V. Kovalevskaya (according to material in her correspondence). Istor.-Mat. Issled. 7, 666–712 (2 plates) (1954). (Russian)

Depman, I. Ya. On the biography of S. V. Kovalevskaya. Istor.-Mat. Issled. **7**, 713–715 (1954). (Russian)

***Freudenthal, Hans. Leibniz und die Analysis Situs.** Homenaje a Millás-Vallicrosa, Vol. I, pp. 611–621. Consejo Superior de Investigaciones Científicas, Barcelona, 1954.

Rybnikov, K. A. The mathematical manuscripts of Marx. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 197–199 (1955). (Russian)

Prudnikov, V. E. Four letters to M. V. Ostrogradskiĭ. Istor.-Mat. Issled. **7**, 716–719 (1954). (Russian)

Sergescu, Pierre. D. Pompeiu (1873–1954). Enseignement Math. **40** (1951–1954), 70–71 (1955).

Sarymsakov, T. A. Obituary: Vsevolod Ivanovič Romanovskiĭ. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 79–88 (1 plate) (1955). (Russian)
Biography and list of published papers.

Gusev, V. A. The first Russian professor of descriptive geometry, Ya. A. Sevast'yanov. Trudy Inst. Istorii Estest. **4**, 184–194 (1952). (Russian)

***Smirnov, V. I., i Kulyabko, E. S. Mihail Sofronov, russkiĭ matematik serediny XVIII veka. [Mihail Sofronov, a Russian mathematician of the middle of the XVIIIth century.]** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1954. 54 pp. 2.10 rubles.

Järnefelt, Gustaf. Karl Fridhiof Sundman. Soc. Sci. Fenn. Årsbok **30**, no. 2, 13 pp. (1 plate) (1953).

Obláth, Richárd. L'oeuvre mathématique de Gyula Szőkefalvi-Nagy. Mat. Lapok **5**, 189–243 (1954). (Hungarian. Russian and French summaries)

Archibald, R. C. Wolfram, Vega, and Thiele. Math. Tables Aids Comput. **9**, 21 (1955).

FOUNDATIONS

Sueki, Takehiro. On several kinds of logical systems. Rep. Univ. Electro-Commun. **2**, 223–236 (1950). (Japanese)

Sueki, T. Of general properties of logical systems. Rep. Univ. Electro-Commun. **3**, 1–24 (1951). (Japanese. English summary)

Sueki, T. The formulization of 2-valued and n-valued systems. I, II, III. Rep. Univ. Electro-Commun. **4**, 1–24 (1952); **5**, 1–18 (1953); **6**, 1–19 (1954). (Japanese. English summary)

***Asser, Günter. Die endlichwertigen Łukasiewiczschen Aussagenkalküle.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 15–18. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Asser gives a brief exposition of Łukasiewicz's many-valued logics and discusses methods used for their formalisation. He refers to new results which he has obtained and intends to publish. *A. Rose* (Nottingham).

***Reichenbach, Hans. Les fondements logiques de la théorie des quanta. Utilisation d'une logique à trois valeurs.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 103–105; discussion, pp. 105–114. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

This begins with a short statement by Reichenbach that the use of the three-valued logic of Łukasiewicz in quantum mechanics permits the description of unobserved events without paradox because of the availability of the third truth value "undecided", in addition to "true" and "false". There then follows a report of the discussion that ensued, dealing mostly with the possible role of various logical systems in quantum mechanics. This discussion is quite long and interesting, the participants besides Reichenbach being Bodiou, Feys, Destouches, Mme. Fevrier, Bochenski, Gonseth, and Kurepa. *O. Frink* (University Park, Pa.).

***Bodiou, G. Utilisation du formalisme quantique à la topologie du treillis (non modulaire) des propositions "a posteriori" ou à la modification du concept de "corpuscule localisé".** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 129–132. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Noting that a posteriori propositions of quantum mechanics form a non-modular lattice like that of the closed linear subspaces of Hilbert space, the author seeks a real-valued valuation of this lattice which will have a probability interpretation and will determine a metric topology. He defines one, but notes that the lattice operations are not continuous in the resulting topology. In terms of the metric he defines a notion of localisation of a moving particle in the sense of quantum mechanics. *O. Frink.*

***Robinson, Abraham. L'application de la logique formelle aux mathématiques.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 51–63; discussion, p. 64. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

This paper contains a description of the author's methods and some of his results obtained by applications of extensions of Gödel's completeness theorem to various branches of mathematics. A more detailed version of most of these may be found in A. Robinson, On the metamathematics of algebra [North Holland Publ. Co., Amsterdam, 1951; MR **13**, 715]. *I. Novak Gál* (Ithaca, N. Y.).

***Kreisel, G. Applications of mathematical logic to various branches of mathematics.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 37–49; discussion, p. 50. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Several different applications are discussed. First the author sets up a correspondence between statements $\mathfrak{A}$ about the complex affine plane and statements $\mathfrak{A}'$ about the

complex and real projective plane and gives necessary and sufficient conditions that $\mathfrak{A}'$ should hold in the real and complex plane if $\mathfrak{A}$ holds in the complex affine plane. Next he discusses systems of number-theory with a limited number of primitive recursive equations as axioms and with an induction scheme such as

$$\frac{A(0), A(n)\rightarrow A(n')}{A(n)} \text{ or } \frac{A(0), A(n)\leftrightarrow A(n')}{A(n)}.$$

The need for the introduction of auxiliary functions is discussed. The author defines a direct proof as one which contains no function symbols other than the ones contained in the theorem itself or in the set of defining equations for these functions. Lastly there is a discussion and applications of systems of number-theory which satisfy the ω-condition: If $(Ex)A(x)$ is proved and $A(x)$ is general recursive, then $A(O^{(n)})$ holds for some integer $O^{(n)}$. *I. Novak Gál.*

Izumi, Yosihisa. Sur les formes normales. Tôhoku Math. J. (2) **6**, 26–29 (1954).

Let A be any formula, $R(A)$ another formula such that $R(A)$ is satisfied if and only if A is satisfied, $D(A)$ another formula such that $D(A)$ is provable if and only if A is provable. Then A is satisfiable if and only if $\sim D(\sim A)$ is satisfiable, and A is provable if and only if $\sim R(\sim A)$ is provable. *I. Novak Gál* (Ithaca, N. Y.).

Takeuti, Gaisi. Construction of the set theory from the theory of ordinal numbers. J. Math. Soc. Japan **6**, 196–220 (1954).

The author shows that within the theory of ordinals it is possible to construct a model of von Neumann set-theory. The methods of Gödel [The consistency of the continuum hypothesis, Princeton, 1940; MR **2**, 66] are used. *I. Novak Gál* (Ithaca, N. Y.).

Goodstein, R. L. The relatively exponential, logarithmic and circular functions in recursive function theory. Acta Math. **92**, 171–190 (1954).

The author considers the usual approximations of the elementary functions of analysis by finite sums, and states conditions on these sums which, in the language of informal analysis, ensure that the corresponding limit functions are constant, continuous, uniformly differentiable etc. By establishing the appropriate conditions he proves the analogues of the fundamental theorems on the elementary functions; e.g. if $E(n, x)$ denotes $\sum_0^n x^r/r!$, he finds a recursive $N(k, x, a)$, for integral k and rational x and a, such that,

(i) $|E(n, x)E(n, a)-E(n, x+a)| < 10^{-k}$ for $n \geqq N(k, x, a)$;

this is the analogue of the multiplicative property of $\exp(x)$. The main aim of all this is to keep the proofs within recursive arithmetic. It is not surprising that no new ideas are introduced. For, if real numbers are represented in one of the standard ways, the special functions of analysis with rational arguments can be represented in Peano's arithmetic (P); e.g. the nth decimal approximation of $\exp(x)$ is given by

$$10^{-n}\mu_p(Ek_1)(Ek_2)(m)[m \geqq k_1 \rightarrow .E(m, x)+k_2^{-1} < (p+1)10^{-n}];$$

with the appropriate definitions of multiplication and equivalence of decimals, $\exp(a)\cdot\exp(x) \equiv \exp(x+a)$ can be proved in (P), and so can the convergence of $E(n, x)$ to $\exp(x)$ with a recursive modulus of convergence: hence (i) can be proved in (P). Since (i) is quantifier-free, the quantifiers can be eliminated from the proof by means of a systematic method [cf. Kreisel, British J. Philos. Sci. **4**, 107–129 (1953); **357** (1954), p. 123; MR **15**, 670]. Further, if free function variables (with integral arguments) and the appropriate substitution rules are added to (P), the usual special functions of analysis can be defined not only for rational, but also for real arguments, and the implications of elementary informal analysis can be established in the form: if the premise (with a free function variable) can be proved, so can the conclusion. In particular, differentiable, but not uniformly differentiable functions, introduced via Cantor's ternary set, can be treated, although they are expressly excluded by the author. In this way the whole elementary theory of convergence, completeness, and compactness can be used for proofs of quantifier-free formulae, and then eliminated (as in loc. cit.) to provide the recursive proofs which the author is after. *G. Kreisel* (Reading).

*Dequoy, N. **Axiomatique intuitionniste sans négation de la géométrie projective.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1955. 108 pp. 1250 francs.

The author begins by giving an account of the fundamental ideas of intuitionism and of the negationless intuitionism of Griss [Nederl. Akad. Wetensch., Proc. **49**, 1127–1133 (1946); MR **8**, 307]. The main rules of deduction which she uses for propositions are as follows, "$p \rightarrow q$" being interpreted as "from p follows q":

$$p\&q\rightarrow q\&p,\quad p\&q\rightarrow p,\quad p\&q\rightarrow q,\quad (p\rightarrow q)\&(q\rightarrow r)\rightarrow(p\rightarrow r),$$
$$(p\rightarrow q)\rightarrow(p\&r\rightarrow q\&r),\quad (p\rightarrow q)\&(r\rightarrow s)\rightarrow(p\&r\rightarrow q\&s),$$
$$(p\rightarrow r)\rightarrow(p\&q\rightarrow r),\quad (p\rightarrow q)\&r\rightarrow(p\rightarrow q\&r).$$

Only realisable suppositions are to be made. Species are denoted by $a, b, c, \cdots$ and the alternation symbol is applied to these only. The rules for species are:

$$a\vee b\rightarrow b\vee a,\quad (a\rightarrow b)\rightarrow(a\vee c\rightarrow b\vee c),$$
$$(a\rightarrow b\vee c)\&(b\rightarrow d)\&(c\rightarrow e)\rightarrow(a\rightarrow d\vee e).$$

The author then constructs analytical geometry, first giving an account of the theory of real numbers of Griss [ibid. **54**, 193–199 (1951); MR **13**, 310] and stating certain axioms for skew fields. In the latter she uses the symbol # which denotes a special form of inequality.

In the rest of the book the above work is applied to the study of plane and solid geometry and to the theory of order in geometry. *A. Rose* (Nottingham).

Denbow, C. H. Postulates and mathematics. Amer. Math. Monthly **62**, 233–236 (1955).

Menger, Karl. On variables in mathematics and in natural science. British J. Philos. Sci. **5**, 134–142 (1954).

By a variable quantity the author means a numerical-valued single-valued function of a general variable. By a function he means a numerical-valued single-valued function of a numerical variable. He considers what is meant by one variable quantity being a function of another variable quantity. He states that in many cases where the term "variable" is used in applications of mathematics to the physical sciences, the notion actually involved is that of variable quantity in his sense, rather than the concept mathematicians have associated with the term variable.

In dealing with relations between functions as one does in the calculus, if the symbol I for the identity function is introduced, then it becomes possible and also desirable to suppress the symbol x for the independent variable in all formulas. The author claims that in addition to the simplification which thereby results, the shift in emphasis from

the non-essential to the essential promotes a clearer understanding of the true situation. *O. Frink.*

Menger, Karl. Variables de diverses natures. Bull. Sci. Math. (2) **78**, 229–234 (1954).

The author maintains that one can and should dispense with variables like the letter x in function theory, for example, by writing cos and log in place of $\cos x$ and $\log x$, and by representing the power function by I^n instead of x^n (where I stands for the identity function and is not a variable). This leaves the term variable free for other uses. He suggests the term "variable quantity" to represent a real-valued function of any variable, the term "function" being restricted to real-valued functions of a real variable. Familiar physical quantities such as temperature and pressure are claimed to be actually variable quantities in the author's sense. Adoption of this system would simplify formulas and also give a clearer understanding of the relation of function theory to its applications. *O. Frink.*

Rényi, Alfréd. Basic problems of the calculus of probabilities in the light of dialectical materialism. Časopis Pěst. Mat. **79**, 189–218 (1954). (Czech)

Translation of a paper published in Hungarian in A Filozófiai Évkönyv **1952**, 63–97.

Lorent, Henri. Sur l'"indéfiniment" mathématique. Enseignement Math. **40** (1951–1954), 47–56 (1955).

ALGEBRA

Parker, W. V. A note on a theorem of Roth. Proc. Amer. Math. Soc. **6**, 299–300 (1955).

Roth [same Proc. **5**, 1–3 (1954); MR **15**, 672] considered two square matrices A, B such that rank $(A-B)\leqq 1$, and with respective characteristic polynomials $\alpha_0(x^2)-x\alpha_1(x^2)$, $\beta_0(x^2)-x\beta_1(x^2)$; he found the characteristic polynomial of AB. In the present paper, the result

$$\det(xI-AB-k(A-B)) = (-1)^n\{\alpha_0\beta_0-x\alpha_1\beta_1+k\alpha_0\beta_1-k\alpha_1\beta_0\}$$

is established. If the decomposition R^*S of $A-B$ as a product of $n\times 1$, $1\times n$ matrices is known, also the characteristic functions of $AB+c(A-B)^*$ can be found

$$[k=c(SR^*)^{n-1}].$$

J. L. Brenner (Pullman, Wash.).

Goddard, L. S. On the characteristic function of a matrix product. Proc. Amer. Math. Soc. **6**, 296–298 (1955).

A second proof is given of the theorem of Roth cited in the preceding review. Generalization: let A_i be $n\times n$;

$$\det(xI-A_i)=\alpha_{0i}(x^3)+x\alpha_{1i}(x^3)+x^2\alpha_{2i}(x^3) \quad (i=1, 2, 3);$$
$$H=A_1+A_2+A_3; \quad K=A_1A_2+A_1A_3+A_2A_3;$$
$$\text{rank } (xH-K)\leqq 1.$$

Then

$$\det(xI-A_1A_2A_3)=\alpha_{01}\alpha_{02}\alpha_{03}+x\alpha_{11}(\alpha_{02}\alpha_{23}+\alpha_{03}\alpha_{22}) +x\alpha_{12}(\alpha_{01}\alpha_{23}+\alpha_{03}\alpha_{21})+x\alpha_{13}(\alpha_{01}\alpha_{22}+\alpha_{02}\alpha_{21}) +x^2\alpha_{21}\alpha_{22}\alpha_{23}, \quad \alpha_{ij}=\alpha_{ij}(x).$$

J. L. Brenner (Pullman, Wash.).

Epstein, Marvin, and Flanders, Harley. On the reduction of a matrix to diagonal form. Amer. Math. Monthly **62**, 168–171 (1955).

According to L. E. Dickson [Algebras and their arithmetics, Univ. of Chicago Press, Chicago, 1923, p. 108] an algebra $\mathfrak{A}$ with basis u_i $(i=1, \cdots, n)$ over a field of characteristic zero is semi-simple if and only if the matrix of traces u_iu_j $(i, j=1, \cdots, n)$ is non-singular. This paper discusses the special case in which $\mathfrak{A}$ is the algebra of all polynomials in a square matrix A, so that semi-simplicity of $\mathfrak{A}$ is equivalent to similarity of A to a diagonal matrix in an extension field. No references are given. *M. F. Smiley.*

Wade, Bailey T. The alternant matrix. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Tech. Note No. **933**, 9 pp. (1954).

Certain polynomial interpolation problems require the inverse of the alternant matrix $A=\|(a_i)^{j-1}\|$ of order n. Since machine methods of inverting A are likely to have scaling difficulties, the author recommends using an explicit formula for the inverse. He gives expressions taken from A. C. Aitken, Determinants and matrices, [Oliver and Boyd, Edinburgh, 1951 edition; see MR **1**, 35 for a review of the 1939 edition]. *G. E. Forsythe.*

Mauldon, J. G. Composite matrices. Quart. J. Math., Oxford Ser. (2) **6**, 80 (1955).

The results of S. N. Afriat [same J. (2) **5**, 81–98 (1954); MR **16**, 105] and Egervary [Acta Sci. Math. Szeged **15**, 211–222 (1954); MR **16**, 327] are shown to hold not merely over a field, but over any commutative ring. For example, if F, A, B, C, D are square matrices, $F=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $\det F=\det(AD-BC)$, provided A, B, C, D commute.

J. L. Brenner (Pullman, Wash.).

Sherman, Seymour. Doubly stochastic matrices and complex vector spaces. Amer. J. Math. **77**, 245–246 (1955).

The author proves the following converse of a result of A. Horn [Amer. J. Math. **76**, 620–630 (1954); MR **16**, 105]. Let $x=(x_i)$, $y=(y_i)$ be two complex n-vectors, and suppose that, for arbitrary complex $c_1, c_2, \cdots, c_n$, the point $\sum_{i=1}^n c_iy_i$ of the complex plane lies in the convex hull of the set of points $\sum_{i=1}^n c_ix_{\alpha(i)}$, where α runs over all permutations of the numbers $\{1, 2, \cdots, n\}$. Then $y=Px$, where P is a doubly stochastic matrix, i.e.,

$$P_{ij}\geqq 0, \quad \sum_{i=1}^n P_{ij}=1, \quad \sum_{j=1}^n P_{ij}=1.$$

F. Smithies (Cambridge, England).

Fan, Ky, and Hoffman, A. J. Some metric inequalities in the space of matrices. Proc. Amer. Math. Soc. **6**, 111–116 (1955).

Let $\|A\|$ be any unitarily invariant norm in the vector space of $n\times n$ matrices over the complex field, i.e., such that $\|A\|=\|UA\|=\|AU\|$ for every unitary matrix U. The authors prove the following inequalities. (i) If $A=UH$, where U is unitary and H is non-negative definite, and W is also unitary, then

$$\|A-U\|\leqq\|A-W\|\leqq\|A+U\|.$$

This is analogous to the statement that if z $(\neq 0)$ is a complex number, the nearest and farthest points of the unit circle are $\operatorname{sgn} z=z/|z|$ and $-\operatorname{sgn} z$.

(ii) If H is Hermitian, then

$$\|A-\tfrac{1}{2}(A+A^*)\| \leqq \|A-H\|.$$

(The point of the real axis nearest to z is $\Re z$.)

(iii) If H and K Hermitian, and U and V are their respective Cayley transforms, i.e.,

$$U=(H-iI)(H+iI)^{-1}, \quad V=(K-iI)(K+iI)^{-1},$$

then

$$\|U-V\| \leqq 2\|H-K\|.$$

Furthermore, if (α_ν) and (β_ν) are the singular values of $\frac{1}{2}(U-V)$ and $H-K$ respectively, each sequence being arranged in decreasing order, then $\alpha_\nu \leqq \beta_\nu$ $(1 \leqq \nu \leqq n)$.

F. Smithies (Cambridge, England).

Wielandt, Helmut. An extremum property of sums of eigenvalues. Proc. Amer. Math. Soc. **6**, 106–110 (1955).

Let A be a Hermitian operator in n-dimensional unitary space R_n, having eigenvalues $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$. Let S be a subset of the set $\{1, 2, \cdots, n\}$, and let $R=(R_\sigma)_{\sigma \in S}$ be a family of subspaces of R_n such that $\dim R_\sigma = \sigma$, $R_\sigma \subset R_\tau$ $(\sigma \leqq \tau)$. Let $A_R = \min\{\sum_{\sigma \in S}(Ax_\sigma, x_\sigma) : x_\sigma \in R_\sigma, (x_\sigma, x_\tau)=\delta_{\sigma\tau}\}$. Then the author shows that $\sum_{\sigma \in S}\alpha_\sigma = \max_R A_R$. A dual result is also given.

The author then deduces the following theorem. Let A, B and C be Hermitian operators such that $A+B=C$. Let (α_ν), (β_ν), (γ_ν) be their respective sequences of eigenvalues, each arranged in decreasing order. Let S be a subset of $\{1, 2, \cdots, n\}$ containing k elements. Then

$$\sum_{\sigma \in S}\gamma_\sigma \leqq \sum_{\sigma \in S}\alpha_\sigma + \sum_{\sigma=1}^{k}\beta_\sigma.$$

This is shown to be equivalent to a statement by V. Lidskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **75**, 769–772 (1950); MR **12**, 581]: if α, β, γ are the points (α_ν), (β_ν), (γ_ν) and the above assumptions hold, then γ lies in the convex cover of the $n!$ points $\alpha+P\beta$, where P runs over all $n \times n$ permutation matrices. *F. Smithies* (Cambridge, England).

Magnus, Wilhelm. A Fourier theorem for matrices. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **BR-8**, i+19 pp. (1954).

Consider the mapping $u=\exp(iH)$ of the space S_H of $n \times n$ hermitian matrices onto the space S_U of $n \times n$ unitary matrices. Let λ_μ $(\mu=1, \cdots, n)$ be the eigenvalues of H in S_H. Let S denote the region of S_H defined by $|\lambda_\mu - \lambda_\nu| \leqq 2\pi$ $(\mu, \nu=1, \cdots, n)$. Finally, let a measure be defined on S by $d\tau(H)=d\omega(U)$, where $d\omega(U)$ is Haar measure on S_U. The following results are then valid. If (1) $F(t)=(f_{\mu\nu}(t))$ is an $n \times n$ matrix depending on the real parameter t, and (2) $\int_{-\infty}^{\infty}|f_{\mu\nu}(t)|dt < \infty$ $(\mu, \nu=1, \cdots, n)$, then (3) $\int_{-\infty}^{\infty} F(t)\exp(itH)dt = G(H)$ exists for each H in S, and there exists a positive constant L_n, depending only on n, such that (4) $\int_S G(H)\exp(-itH)d\tau = L_n F(t)$, and (5) $\operatorname{trace}\int_{-\infty}^{\infty} F^*(t)F(t)dt = L_n^{-1}\operatorname{trace}\int_S G^*(H)G(H)d\tau$. The proof given is computational and depends on the spectral theorem for hermitian matrices and the classical 1×1 Fourier and Plancherel theorems. *R. Steinberg* (Princeton, N. J.).

Rutishauser, Heinz. Une méthode pour la détermination des valeurs propres d'une matrice. C. R. Acad. Sci. Paris **240**, 34–36 (1955).

The following unusual theorem is stated: Let A be a real symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n \geqq 0$, and with no eigenvector orthogonal to a coordinate axis. Let $A_0=A$. For each $k=0, 1, 2, \cdots$, find the (unique) Cholesky decomposition $A_k=L_kR_k$, where L_k is a lower triangle and R_k is the transpose of L_k. Define $A_{k+1}=R_kL_k$. Conclusion: $\lim A_k$ is the diagonal matrix

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

G. E. Forsythe (Los Angeles, Calif.).

de Bruijn, N. G., and Szekeres, G. On some exponential and polar representations of matrices. Nieuw Arch. Wisk. (3) **3**, 20–32 (1955).

The authors employ the functional calculus of Fantappié [C. R. Acad. Sci. Paris **186**, 619–621 (1928)] to obtain a unified treatment of polar representations of complex matrices. Since many of the very numerous results are known, we cite only the following one: Every non-singular complex matrix is the product of a matrix B and a real matrix, where B satisfies $B^{-1}=\bar{B}$. *M. F. Smiley* (Stanford, Calif.).

Fantappiè, Luigi. Sulle funzioni di una matrice. An. Acad. Brasil. Ci. **26**, 25–33 (1954).

Il s'agit essentiellement d'une méthode pratique pour le calcul des fonctions d'une matrice carrée K d'ordre fini, d'après la formule

$$g(K)=\frac{1}{2\pi i}\int_C \frac{g(\lambda)}{\lambda I - K}d\lambda$$

dont l'emploi, justifié jadis par l'auteur [C. R. Acad. Sci. Paris **186**, 619–621 (1928)] exige seulement que $g(\lambda)$ soit une fonction de la variable complexe localement analytique sur le spectre de K. Soit $p(\lambda) \equiv \lambda^\nu + b_1\lambda^{\nu-1} + \cdots + b_{\nu-1}\lambda + b_\nu$ un polynôme quelconque vérifiant $p(K)=0$. Si l'on exprime $[p(\lambda)]^{-1}$ sous la forme $\sum_{ji}(\lambda-\rho_j)^{-l-1}$ et si l'on pose $q_1(\alpha)=\alpha+b_1$, $q_r(\alpha)=\alpha q_{r-1}(\alpha)+b_r$ $(r=1, \cdots, \nu-1)$, on a $g(K)=\sum_1^\nu g_rK^{r-1}$, avec

$$g_r=\sum_{ji}\frac{h_{ji}}{l!}\left\{\frac{d^l}{d\lambda^l}[q_{r-1}(\lambda)g(\lambda)]\right\}_{\lambda=\rho_j}.$$

J. Sebastião e Silva (Lisbonne).

Del Pasqua, Dario. Sul calcolo delle matrici. Collect. Math. **6**, 117–123 (1953).

L'auteur étudie le rôle de l'équation minime d'une matrice K dans le calcul des fonctions $f(K)$ de K suivant la formule intégrale de Cauchy [cf. l'analyse précédente]. Soient $\alpha_1, \cdots, \alpha_\nu$ les racines de cette équation, avec les ordres $e_1, \cdots, e_\nu$. En observant que chaque élément f_{rs} de la matrice $f(K)$ est de la forme

$$\text{(a)} \qquad f_{rs}=\sum_{i=1}^{\nu}\sum_{j=1}^{e_i}c_{ij}^{rs}f^{(j)}(\alpha_i),$$

où les coefficients c_{ij}^{rs} ne dépendent que de la matrice K; l'auteur propose une définition directe de $f(K)$, au moyen de (a). Cette définition serait plus générale que celle de Fantappiè, tout en vérifiant les mêmes conditions.

J. Sebastião e Silva (Lisbonne).

Šestakov, V. I. On transformation of a monocyclic sequence into a recurrent one. Dokl. Akad. Nauk SSSR (N.S.) **98**, 541–544 (1954). (Russian)

The author considers a sequence of n-digit binary integers

$$\eta(0), \eta(1), \eta(2), \cdots$$

which are ultimately periodic but are subject to the conditions that the initial non-periodic part together with the proper period consist of distinct integers. Such a sequence is produced when

$$\eta(i+1)=\phi(\eta(i)),$$

where ϕ is a single-valued function whose values are integers $<2^n$ (a condition not explicitly stated by the author). Conversely, any sequence η of the above type determines such a function ϕ. Mention is made of an application to a system of n relays presumably of a digital computer. If the ith relay is energized at the jth program step then we may take the ith digit of $\eta(j)$ to be 1, otherwise it is zero. The system of relays is thus equivalent to a function ϕ. The author states that his theory extends to the case of $n\to\infty$.

D. H. Lehmer (Berkeley, Calif.).

Šestakov, V. I. An algebraic method of systhesis of multi-step relay systems. Dokl. Akad. Nauk SSSR (N.S.) **99**, 987–990 (1954). (Russian)

The author is concerned with the design of relay systems presumably for a digital computer. He uses a vector representation method in which the components are elements of a Boolean algebra and the three operations

$$u'=1-u,$$
$$u\cdot v=uv,$$
$$u\oplus v=u+v \pmod 2$$

applied termwise to these vectors to construct characteristic functions for the expansion of a desired relay function. The treatment is only general and full of obscurities. Reference is made to the paper reviewed above. *D. H. Lehmer.*

Robinson, G. de B., and Thrall, R. M. The content of a Young diagram. Michigan Math. J. **2** (1953–1954), 157–167 (1955).

A partition of n, $[\lambda]\equiv[\lambda_1,\cdots,\lambda_r]$, is associated with a graph of n nodes of which λ_i are in the ith row, in the first λ_i columns in this row. If the node in the ith row, jth column is associated with the number $i-j$, consider the aggregate of n numbers associated with n nodes. If this aggregate includes the integer h repeated μ_h times, the content of the partition $[\lambda]$ is defined as the expression $C[\lambda]=\prod x_h^{\mu_h}$, the number h extending over negative zero and positive integers. The content of a partition defines the partition unequally.

An operator S is defined, operating from the right, such that $x_hS=x_{h+1}$. An expression $\theta=\prod x_h^{\mu_h}$ will be the content of some partition if

$$m_h=\mu_h-\mu_{h+1}\begin{cases}0,1 & \text{for } h\geqq 0,\\ 0,-1 & \text{for } h<0.\end{cases}$$

Also $\sum m_h=0$. The expression $\theta/\theta S=\prod x_h^{m_h}$ is called the trace of the partition and is denoted by $D[\lambda]$.

In the modular case the number $i-j$ is replaced by its least positive residue (mod q). The content then becomes the q-content, $C_q[\lambda]=\prod_{h=0}^{q-1} y_h^{\mu_h}$, and the trace becomes the q-trace

$$D_q[\lambda]=C_q[\lambda]/C_q[\lambda]S=\prod y_h^{\mu_h-\mu_{h+1}}=\prod y_h^{m_h}.$$

The q-content does not determine the partition uniquely. Two partitions have the same q-content if they are partitions of the same numbers and if they have the same q-core. Two partitions have the same q-trace if and only if they have the same q-core. Given an expression

$$y_0^{m_0}y_1^{m_1}\cdots y_{q-1}^{m_{q-1}}$$

such that the m_i are integers and $\sum m_i=0$, there is exactly one q-core of which t is the q-trace.

Various connections are obtained between the indices m_i and the properties of the partition of the q-core. The following generating function is obtained

$$(1-Sx_0)^{-1}(1-Sx_0x_1)^{-1}(1-Sx_0x_1x_2)^{-1}\cdots=1+\sum C[\lambda]$$

summed for all partitions of all integers. The extension to the modular case is immediate. *D. E. Littlewood.*

Sce, Michele. Monogeneità e totale derivabilità nelle algebre reali e complesse. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 30–35 (1954).

The author's aim is to determine those associative algebras A for which an analogue of Cauchy's theorem holds, i.e. every differentiable function gives zero if integrated around a closed hypersurface (bounding a closed region where the function is differentiable). The obvious definitions of right and left differentiability reduce to the conditions that the Jacobian matrix of the function

$$f(z)=\sum f_i(x_1,\cdots,x_n)e_i$$

(e_i=basis elements of A) should be in the range of the right or left regular representation of A. The definition of right (left) monogenic functions (right (left) integrals vanish over $(n-1)$-cycles) reduces to the condition $\sum e_ie_j\partial f_i/\partial x_j=0$ ($\sum e_je_i\partial f_i/\partial x_j=0$), [cf. Kriszten, Comment. Math. Helv. **26**, 6–35 (1952); MR **15**, 213]. The author points out that the latter definition is not invariant under the full linear group, although the former is (this is apparently due to the fact that the definition of the integral depends on the inner product). He thus defines A to be a solenoidal algebra if there is a basis (i.e. an inner product) such that every right differentiable function is right monogenic, and a bisolenoidal algebra if there is a basis such that every right differentiable function is left (hence also right) monogenic. He then shows that the direct sum and tensor product of bisolenoidal algebras is bisolenoidal (indeed, only one factor need be bisolenoidal). *B. Crabtree* (Durham, N. H.).

Sce, Michele. Monogeneità e totale derivabilità nelle algebre reali e complesse. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 188–193 (1954).

In this note the solenoidal algebras [see the preceding review] of orders 2, 3, and 4 with either real or complex scalars are determined. The author also notes that every total matric algebra and all real or complex Clifford algebras are solenoidal but not bisolenoidal. Further, by appealing to the Wedderburn structure theorems he shows that every real semi-simple algebra having no one-dimensional simple component is solenoidal (but that those of orders 1, 3, 5, 7 are not) and that a necessary condition for an algebra A with radical R to be solenoidal is that $A-R$ be solenoidal. The remainder of the paper contains many explicit results concerning which orders an algebra may have if it is solenoidal. *B. Crabtree* (Durham, N. H.).

Abstract Algebra

Lowig, H. F. J. Gesetzrelationen über frei erzeugten Algebren. J. Reine Angew. Math. **193**, 129–142 (1954).

The present paper follows from an earlier paper [J. Reine Angew. Math. **190**, 65–74 (1952); MR **14**, 443] and is in the same highly abstract notation. Like that paper it has

as its aim the avoidance of logical difficulties implicit in the definitions of various concepts connecting with free algebras given by G. Birkhoff [Proc. Cambridge Philos. Soc. 31, 433–454 (1935)]. Löwig's precise formulations enable one to decide, for example, whether or not the algebra over A with no operations is the same as the algebra over A all of whose operations are functions on no elements. He gives safe proofs of a number of the results from Birkhoff's paper, including, for example, that the set of all laws of a given algebra form a complete lattice; and that the algebra over a certain set A obeying only the laws of a certain algebra is isomorphic to a subalgebra of the direct product of subalgebras generated by maps of A into algebras of a given set. *H. A. Thurston* (Bristol).

Jakubík, Ján. Congruence relations on abstract algebras. Czechoslovak Math. J. 4(79), 314–317 (1954). (Russian. English summary)

It is shown by an example that there exists an algebra A with a one-element subalgebra S such that all congruence relations on A are permutable and there exist distinct congruence relations on A with the class containing S being the same in R and R'. The example consists of a multiplicative group with at least two elements, with zero adjoined; $S=\{0\}$, R is the partition into single-element classes, and $R'=\{\{0\}, G\}$. This solves Problem 33 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR 10, 673; for another solution, see A. I. Mal'cev, Mat. Sb. N.S. 35(77), 3–20 (1954); MR 16, 440]. In another example, S is a two-sided unit. *P. M. Whitman* (Silver Spring, Md.).

Jakubík, Ján. System of congruence relations on lattices. Czechoslovak Math. J. 4(79), 248–273 (1954). (Russian. English summary)

Given a lattice S and subsets A and B thereof, denote by $R_1(A)$ the minimal partition of the elements of S which is determined by a congruence relation on S and which has all elements of A in the same class of the partition. Properties of such partitions are studied. If A and B are convex sublattices of S, consider the system, given u and v,

$$\text{(I)} \qquad x \equiv u \bmod R_1(A), \quad x \equiv v \bmod R_1(B).$$

The following condition is necessary for (I) to have a solution:

$$\text{(II)} \qquad u \equiv v \bmod R_1(A) \cup R_1(B).$$

It is proved that if S is distributive, then S is relatively complemented if and only if (II) implies that (I) has a solution; other similar conditions are also found. These generalize the Chinese Remainder Theorem of number theory, studied by V. K. Balachandran [J. Indian Math. Soc. (N.S.) 13, 76–80 (1949); MR 11, 309] for A and B ideals of a lattice.

For modular lattices, sufficient conditions for existence of a solution of (I) are found by considering ideals in S', the minimal distributive lattice of classes of a congruence relation of S. Also, a theorem of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 77; MR 10, 673], that congruence relations on a modular lattice correspond one-to-one to the sets of classes of projective prime quotients which they annul (and hence form a Boolean algebra), is generalized from finite lattices to discrete lattices (i.e. in which bounded chains are finite). *P. M. Whitman* (Silver Spring, Md.).

Lorenzen, Paul. Über die Korrespondenzen einer Struktur. Math. Z. 60, 61–65 (1954).

By a "structure" the author means the set of all systems isomorphic to an algebra M with finitary operations. A binary relation ρ in M is called a correspondence if it is isotone with respect to all the operations of M. Correspondences thus include congruence relations as well as endomorphisms. In the set of all correspondences in M there are defined the usual operations of meet $\rho \cap \sigma$, which is the intersection, relative product $\rho\sigma$, which is in general non-commutative, and converse. The author neglects to define a join operation for correspondences, and hence treats the system of all correspondences in M as a semilattice with an additional non-commutative product operation replacing the join operation of lattice theory. (Since the intersection of an arbitrary collection of correspondences is a correspondence, it would seem that the system of all correspondences in M is a complete lattice. Hence a join operation could be defined and studied if that were convenient.)

The author lists a number of simple properties of the operations of meet, product, and converse of correspondences, and then adopts an abstract viewpoint. He calls any system with three operations obeying these rules a "Bund", and after defining the concept "ρ is normal relative to σ", proves an analog of the Zassenhaus refinement theorem for these abstract systems. *O. Frink.*

Sasaki, Usa. Orthocomplemented lattices satisfying the exchange axiom. J. Sci. Hiroshima Univ. Ser. A. 17, 293–302 (1954).

The author studies lattices which are complete, relatively atomic, ortho-complemented, satisfy MacLane's exchange axiom, and in addition have the property that orthogonal elements always form a modular pair. This type of lattice forms a generalization of the lattices of all closed linear subspaces of a (not necessarily separable) Hilbert space. They differ from the exchange lattices of MacLane (called matroid lattices by Maeda) in not being necessarily upper continuous. Although not always modular, they are self-dual, and satisfy also the duals of various exchange axioms.

The author shows that every element has an orthogonal basis of points (atoms), and that every orthogonal basis of an element has the same cardinal number if an additional condition holds, namely that a point contained in a join of points is contained in the join of a countable subset. Each such lattice, if reducible, is a direct sum of lattices of the same type. Intervals form sublattices of the same type. A kind of mapping called a projection onto an element is studied, which reduces to an orthogonal projection in the Hilbert-space case. *O. Frink* (University Park, Pa.).

Kowalsky, Hans-Joachim. Distributivität in atomaren Booleschen Verbänden. Arch. Math. 6, 9–12 (1954).

The theorem that a Boolean algebra B is atomic if and only if it is completely distributive in the wider sense is given a direct proof based on a discussion of the class of linearly ordered subsets of $B-\{0\}$ which are maximal in the downward direction. *L. H. Loomis.*

Jónsson, Bjarni. Modular lattices and Desargues' theorem. Math. Scand. 2, 295–314 (1954).

The author studies in greater detail the connection between a complemented modular lattice B and the (generalized) projective space which it determines. It was shown by the reviewer that B is isomorphic to a sublattice of the lattice A of all subspaces of a certain projective space

[Trans. Amer. Math. Soc. 60, 452–467 (1946); MR 8, 309]. It is shown here that every lattice identity which holds in B holds in A also. More generally, this is true for lattices A and B whenever A is a perfect extension of B, in which case B is said to be regular in A, a relationship which the author defines and investigates.

It was shown by Schützenberger and the author that the Desargues theorem for a projective space M is equivalent to any of several similar lattice identities in the lattice of subspaces of M. A lattice in which one of these identities holds is called by the author Arguesian. It is shown that this Arguesian identity implies the modular law.

Using previous results of the author, it is shown that for complemented modular lattices B the following conditions are equivalent: (1) B is Arguesian. (2) B is isomorphic to a lattice of commuting equivalence relations. (3) B is isomorphic to a lattice of normal subgroups of a group. (4) B is isomorphic to a lattice of subgroups of an abelian group. (5) B is isomorphic to a lattice of subspaces of an Arguesian projective space.

However, for lattices which are merely modular but not complemented, it is shown that the five conditions are not equivalent. In particular he constructs a counterexample which is five-dimensional, Arguesian, and isomorphic to a lattice of commuting equivalence relations, but is not isomorphic to a lattice of normal subgroups of a group, nor of subspaces of a projective space. This lattice may be finite as well as finite-dimensional. *O. Frink.*

Ellis, David. Some saddle-points in $\Lambda\otimes\overline{\Lambda}$. Publ. Math. Debrecen 3 (1953), 168–170 (1954).

Properties p (q) of elements of a complete lattice are said to be lower (upper) properties if they are shared with all predecessors (successors) of an element. An element is a saddle-point with respect to lower and upper properties p and q if it is maximal with respect to having property p, and minimal with respect to having property q. For the lattice of all T_1 topologies on an infinite set S, Vaidyanathaswamy [Treatise on set theory, part I, Indian Math. Soc., Madras, 1947; MR 9, 367] showed in effect that the property of a topology of being T_2 and bicompact is a saddle-point property. The author shows that if f is a permutation of S, then the properties that f is a continuous mapping, and that f is an open mapping, with respect to a topology on S, are a pair of saddle-point properties of the lattice of all T_1 topologies on S. *O. Frink* (University Park, Pa.).

Ellis, David. Remarks on Boolean functions. J. Math. Soc. Japan 5, 345–350 (1953).

A remark that the Boolean functions of one variable in a Boolean algebra mapping it onto itself are the autometrized motions. *L. Nachbin* (Rio de Janeiro).

Bagley, R. W. On the characterization of the lattice of topologies. J. London Math. Soc. 30, 247–249 (1955).

Let Λ be the lattice of T_1 topologies on an infinite set and let Λ_0 be the subset of Λ which consists of the last element of Λ and all the elements which are lattice products of hyperplanes. The author proves that Λ_0 is a full set algebra and is maximal (in Λ) with respect to being uniquely complemented and containing $\alpha\wedge\beta$ whenever it contains α and β. *C. T. Yang* (Princeton, N. J.).

Mitsudo, Fujio. A note on the commutativity of certain rings. J. Sci. Hiroshima Univ. Ser. A. 18, 13–14 (1954).

The paper is concerned with a partial generalization of a theorem of the reviewer's that if in a ring R with center Z, if $x^{n(x)}-x\in Z$ for all $x\in R$ where $n(x)>1$, then R is commutative. The author proves: let R be a ring with unit element, and suppose that $x^{n(x)}-h(x)\in Z$ for all $x\in R$ where $n(x)$ is bounded for all $x\in R$ and where the $h(x)$ run through a finite set of regular center elements; then R is commutative. The proof is modelled after that of the reviewer and the theorem of this paper is proved by reduction to the first-cited theorem. *I. N. Herstein* (Philadelphia, Pa.).

Azumaya, Gorô. Strongly π-regular rings. J. Fac. Sci. Hokkaido Univ. Ser. I. 13, 34–39 (1954).

Call a ring π-regular if for every a there exist n and x (depending on a) such that $a^nxa^n=a^n$; call it right π-regular if for every a there exist n and x with $a^{n+1}x=a^n$. The two concepts are presumably independent, but if the nilpotent elements are of bounded index they are shown to be equivalent. This is done by a series of ingenious arguments, with many side results of interest. Sample: if, in a ring of bounded index, $a^2x=a$, then there exists y with $ya^2=a$.

I. Kaplansky (Chicago, Ill.).

Nagata, Masayoshi. On the theory of Henselian rings. II. Nagoya Math. J. 7, 1–19 (1954).

In this paper the author continues his recent studies on Henselian rings, i.e., rings $\mathfrak{o}$ which are quasi-local and in which decomposition of a polynomial into relatively prime polynomials modulo the maximal prime ideal $\mathfrak{p}$ implies a similar decomposition in $\mathfrak{o}$ [same J. 5, 45–57 (1953); MR 14, 529]. He shows, among other results, that the Henselization $\mathfrak{o}^*$ of an integrally closed quasi-local integral domain $\mathfrak{o}$ will contain $\mathfrak{p}\mathfrak{o}^*$ as a prime ideal if $\mathfrak{o}/\mathfrak{p}$ is integrally closed; otherwise, $\mathfrak{o}^*\mathfrak{p}$ is an intersection of prime ideals. The methods of proof lean on suitable adaptations of Hilbert's theory of decomposition fields, e.g., properties of ring extensions given by polynomials $x^r+d_1x^{r-1}+\cdots+d_r$ with $d_r\in\mathfrak{p}$, d_{r-1} non-$\in\mathfrak{p}$.

O. F. G. Schilling (Chicago, Ill.).

Chevalley, C. La notion d'anneau de décomposition. Nagoya Math. J. 7, 21–33 (1954).

Suppose that $\mathfrak{o}$ and $\mathfrak{J}$ are integral domains with respective quotient fields K and L. Let $\mathfrak{M}$ be a prime ideal of $\mathfrak{J}$ and set $\mathfrak{m}=\mathfrak{M}\cap\mathfrak{o}$. The author terms $\mathfrak{M}$ "controlled" by $\mathfrak{o}$ if (i) $\mathfrak{m}$ generates the maximal ideal in the quotient ring $\mathfrak{J}_{\mathfrak{M}}$ of $\mathfrak{J}$ with respect to $\mathfrak{M}$, and (ii) each coset of $\mathfrak{J}$ modulo $\mathfrak{M}$ contains an element of $\mathfrak{o}$. Using this definition the author proves the existence of an arithmetic decomposition ring $\mathfrak{h}\subset\mathfrak{J}$, $\mathfrak{o}\subset\mathfrak{h}$, which is characterized as the ring for which $\mathfrak{h}\cap\mathfrak{m}$ is controlled by $\mathfrak{o}$. Next, restrictions on the structure of $\mathfrak{o}$ are introduced so that the ring $\mathfrak{h}$ may be shown to coincide with the algebraic decomposition ring of $\mathfrak{M}$, Ω, i.e. the set of invariant elements in $\mathfrak{J}$ with respect to the ordinary decomposition group of prime ideal in the integral closure of a normal extension of L/K. [See also M. Nagata, Nagoya Math. J. 5, 45–57 (1953); MR 14, 529.] For the first step of this assertion the author assumes that $\mathfrak{o}$ is a Noetherian local ring ($\mathfrak{J}$ integral over $\mathfrak{o}$) and shows that $\mathfrak{h}\subseteq\Omega$. Essential parts of the proof are: (i) $\mathfrak{m}$ is the intersection of a finite number of prime ideals of its integral closure in K; (ii) using valuation theory, $\bigcap_{n=1}^{\infty}\mathfrak{m}^n\mathfrak{J}_{\mathfrak{M}}=0$ (no finiteness assumptions are made for $\mathfrak{J}/\mathfrak{o}$). The second step asserting $\Omega\subseteq\mathfrak{h}$ presupposes that $\mathfrak{o}$ and $\mathfrak{J}$ are integrally closed in their respective quotient fields, with L separable

over K. In an appendix the author adds an elegant proof of a theorem Akizuki and Krull stating that a Noetherian integral domain with a single non-trivial prime ideal has an integral closure (in its quotient field) which is in turn Noetherian. *O. F. G. Schilling* (Chicago, Ill.).

Lamprecht, Erich. Bemerkungen zur Struktur des Gruppenringes. Abh. Math. Sem. Univ. Hamburg **19**, 191–197 (1955).

This paper (which is principally expository) offers a proof of the main theorem on symmetric algebras [R. Brauer and C. Nesbitt, Proc. Nat. Acad. Sci. U. S. A. **23**, 236–240 (1937)]. The author concludes by showing that $\mathfrak{G}_K/\mathfrak{n}$ is separable, where $\mathfrak{G}_K$ is the group algebra of a finite group $\mathfrak{G}$ over a field K, and $\mathfrak{n}$ is the radical of $\mathfrak{G}_K$. (Reviewer's note: An elegant proof of the main theorem cited results from trivial simplifications of a proof of T. Nakayama [Ann. of Math. (2) **40**, 611–633 (1939), pp. 619–620; MR **1**, 3].) *M. F. Smiley* (Stanford, Calif.).

Brown, W. P. Generalized matrix algebras. Canad. J. Math. **7**, 188–190 (1955).

A generalized matrix algebra is an algebra A of dimension mn over a field K, with a basis e_{ij}, $i=1, \cdots, m$; $j=1, \cdots, n$, whose multiplication table is given by $e_{ij}e_{pq}=\varphi_{jp}e_{iq}$, $\varphi_{jp} \in K$. (The author assumes K of characteristic 0 but this seems unnecessary for the results.) Such algebras are stated to be related to the algebras ω_f^n, connected with the orthogonal groups [cf. H. Weyl, The classical groups, Princeton, 1939, chap. V, 5; MR **1**, 42]. Every element b in A then corresponds in an obvious way to an $m\times n$ matrix B, and bc corresponds to $B\Phi C$, with $\Phi=(\varphi_{jp})$. By a suitable change of bases, a basis e'_{ij} of A is found with $e'_{ij}e'_{pq}=\varphi'_{jp}e'_{iq}$ and such that $\Phi'=P\Phi Q$, where P, Q are arbitrary non-singular $n\times n$ and $m\times m$ matrices respectively. Thus, it can be assumed that $\Phi=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. It is then easy to see that $A=S+N$ (vector space direct sum), where S is the ordinary $r\times r$ matrix algebra over K and N is the radical of A with $N^2=0$. Furthermore, $N=0$ if and only if A has a unit, in which case $m=n$ and $A=K_n$. *A. Rosenberg* (Evanston, Ill.).

Roşculeţ, Marcel N. Algèbres linéaires non associatives. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 251–262 (1954). (Romanian. Russian and French summaries)

A non-associative linear algebra A_{3^n} having 2^n basis elements is constructed iteratively, generalising complex numbers, quaternions and Cayley numbers which correspond to $n=1, 2, 3$. The norm of an element of A_{3^n} means the sum of the squares of its coefficients, and it is shown, in effect, that the norm of a product of two elements equals the product of their norms only when $n=1, 2, 3$. [For the history of this result and for a more general such iterative process, see Albert, Ann. of Math. (2) **43**, 161–177 (1942); MR **3**, 261.]

The author then discusses three algebras A_{3^n} of orders 3^n ($n=1, 2, 3$) of which A_3, A_9 are associative and A_{27} non-associative. Their basis elements are all cube roots of an identity element; their structural connection is rather like that of the preceding algebras A_{2^n}, and the author defines for each a norm function which has the same multiplicative property. These norms are polynomials of degrees 3, 9, 27 in the coefficients. *I. M. H. Etherington.*

Goldberg, S. I. On the Euler characteristic of a Lie algebra. Amer. Math. Monthly **62**, 239–240 (1955).

The Euler characteristic $\chi(L)$ of a semi-simple Lie algebra L over a field F of characteristic 0 was shown by Chevalley and Eilenberg [Trans. Amer. Math. Soc. **63**, 85–124 (1948); MR **9**, 567] to be zero. This paper proves that $\chi(L)=0$ for any Lie algebra L over any field F. *L. A. Kokoris.*

Herstein, I. N. On the Lie and Jordan rings of a simple associative ring. Amer. J. Math. **77**, 279–285 (1955).

Let A be an associative ring, A^J the Jordan ring obtained by defining multiplication by $a\circ b=ab+ba$, A^L the Lie ring with multiplication $[a, b]=ab-ba$. Clearly, if A^J is simple, then so is A. The author's first theorem is the converse: if A is a simple ring of characteristic $\neq 2$, then A^J is a simple Jordan ring.

The remainder of the paper is devoted to the structure of A^L. Suppose that A has no nonzero locally nilpotent ideals and that $2x=0$ in A implies $x=0$; then any ideal U of A^L which is at the same time a subring of A either is contained in the center of A or contains a nonzero ideal of A. For characteristic $\neq 2$ this implies the principal result of the paper: if A is simple and U an ideal of A^L, then either U is contained in the center of A or U contains $[A, A]$, except in the case that A is 4-dimensional over its center of characteristic 2. The proof for characteristic 2 requires a detailed analysis of its own. *R. D. Schafer* (Storrs, Conn.).

Dieudonné, Jean. Lie groups and Lie hyperalgebras over a field of characteristic $p>0$. II. Amer. J. Math. **77**, 218–244 (1955).

We use the notation and terminology of part I [Comment. Math. Helv. **28**, 87–118 (1954); MR **16**, 12]. For the set $y=(y_1, \cdots, y_n)$ of indeterminates, let R_y denote the right translation operator of an n-dimensional formal Lie group, i.e., $R_y(f(x))=f(\varphi(x, y))$, where φ is the usual abbreviation for the group law. One may write $R_y=\sum_\alpha y_\alpha Z_\alpha$, where the summation is over all exponent systems $\alpha=(\alpha_1, \cdots, \alpha_n)$, $y_\alpha=y_1^{\alpha_1}\cdots y_n^{\alpha_n}$, and the Z_α are certain left-invariant semiderivations of the ring of formal power series in $x=(x_1, \cdots, x_n)$. The Z_α constitute a basis for the hyper Lie algebra of the formal Lie group, and their multiplication table determines the group law. In fact, if $Z_\alpha Z_\beta=\sum_\gamma d_{\alpha\beta\gamma}Z_\gamma$, one has $\varphi_i(x, y)=\sum_{\alpha,\beta} d_{\alpha\beta\epsilon_i}x_\alpha y_\beta$, where $\epsilon_i=(\delta_{1i}, \cdots, \delta_{ni})$. Z_0 is the identity map, whence $d_{\alpha 0\beta}=\delta_{\alpha\beta}=d_{0\alpha\beta}$. One also has $Z_\alpha(x_\beta)=\sum_\gamma d_{\gamma\alpha\beta}x_\gamma$, and $Z_\alpha(fg)=\sum_{0\le\beta\le\alpha} Z_\beta(f)Z_{\alpha-\beta}(g)$. A (non-invariant) characterization of hyper Lie algebras of formal Lie groups is obtained by showing that, conversely, every system of semiderivations Z_α (of a certain height depending on α) with the formal properties just enumerated is actually the system derived from a formal Lie group whose group law is as written above. Moreover, if G and G' are formal Lie groups with hyper Lie algebras $\mathfrak{G}$ and $\mathfrak{G}'$, and u is a homomorphism of $\mathfrak{G}$ into $\mathfrak{G}'$ which sends 1 into 1 and each $\mathfrak{G}_r$ into $\mathfrak{G}_r'$, then u is the derived of a formal group homomorphism of G into G' if and only if $u(Z_\alpha)(f'g')=\sum_{0\le\beta\le\alpha} u(Z_\beta)(f')u(Z_{\alpha-\beta})(g')$, for all power series f' and g'.

The further results concern abelian formal Lie groups over algebraically closed fields. In particular, a complete classification is given for the isomorphism classes of the hyper Lie algebras of the 1-dimensional groups. Each such isomorphism class is shown to contain a certain explicitly described, normalized (in a certain sense) hyper Lie algebra, whose structure depends only on a single parameter $r=0, 1, \cdots$. In this connection, it should be observed that

the author shows elsewhere (to appear in Math. Z.) that each of these hyper Lie algebras actually belongs to a 1-dimensional abelian formal Lie group. Furthermore, it has been proved by M. Lazard [C. R. Acad. Sci. Paris **239**, 942–945 (1954); MR **16**, 219] that every 1-dimensional formal Lie group is abelian.

Now let G be an abelian formal Lie group of arbitrary dimension over an algebraically closed field, and let $\mathfrak{G}_0$ be its Lie algebra (equipped with its pth power map). It is shown that $\mathfrak{G}_0$ can be decomposed into a direct sum $\mathfrak{h}+\mathfrak{k}$, such that the pth power map in $\mathfrak{h}$ is 1-1 and onto, whereas the pth power map is nilpotent in $\mathfrak{k}$. The direct summand $\mathfrak{h}$ is called the core of $\mathfrak{G}_0$, and $\mathfrak{k}$ is called the p-radical of $\mathfrak{G}_0$. It is shown that this decomposition of $\mathfrak{G}_0$ corresponds to a decomposition of G into a direct product $H\times K$, where the Lie algebra of H is isomorphic with $\mathfrak{h}$ and that of K is isomorphic with $\mathfrak{k}$. Moreover, if $\mathfrak{h}$ has dimension m, H is isomorphic with the direct product of m groups W_1^*, the group law of W_1^* being given by $\varphi(x, y)=x+y+xy$. The components H and K are characteristic, in the sense that if $G'=H'\times K'$ is another such decomposed group and if u is a homomorphism of G into G' then u maps H into H' and K into K'.

While the coreless (i.e., $\mathfrak{h}=(0)$) formal Lie groups are not further investigated here, the homomorphisms among the groups of the other type, i.e., direct products of groups W_1^*, are determined in explicit form.

Finally, a number of examples are given to illustrate these notions. In particular, it is shown that the group W_n, derived from the addition of Witt n-vectors, is coreless, and that the group W_n^*, derived from the multiplication of Witt n-vectors, is isomorphic with $W_1^*\times W_{n-1}$.

G. Hochschild (Urbana, Ill.).

Magnus, Wilhelm. On the exponential solution of differential equations for a linear operator. Comm. Pure Appl. Math. **7**, 649–673 (1954).

The first part of the paper gives a proof and applications of a characterization of the Lie elements in a free associative algebra R with n free generators $x_1, \cdots, x_n$ over a field f_0 with zero characteristic applied by Friedrichs in his quantum-mechanical investigations [same Comm. **6**, 1–72 (1953); MR **15**, 80]. An element of R can be uniquely written as a linear combination $F(x_1, \cdots, x_n)$ of power products $x_{v_1}^{m_1}\cdots x_{v_m}^{m_r}$ $(v_1\neq v_2\neq\cdots\neq v_m; m=0, 1, \cdots)$ with coefficients from f_0 or, in an extended sense, as a formal power series. A Lie element of R is an element which can be generated from $x_1, \cdots, x_n$ by the Lie multiplication $[uv]=uv-vu$ and linear combinations. Let R' be an isomorphic replica of R, and $\bar{R}$ the direct product of R and R'. $\bar{R}$ is generated, say in case of two generators $x_1=x$, $x_2=y$ of R, from four generators x, y, x', y' satisfying the conditions $xx'=x'x$, $xy'=y'x$, $yx'=x'y$, $yy'=y'y$. Friedrichs' theorem says: $F(x, y)$ is a Lie element if and only if

$$F(x+x', y+y')=F(x, y)+F(x', y').$$

The author gives a proof of the theorem and applies it to a derivation of the Baker-Hausdorff formula which shows that the unique solution z of the equation $e^xe^y=e^z$ is a Lie element of the free algebra generated by x and y.

The author further investigates a continuous analogue of the Baker-Hausdorff formula: let $A(t)$ be a known function in R of a real parameter t. Assuming that a differentiation with respect to t may be defined obeying the ordinary laws of differentiation, the differential equation $dU/dt=AU$ with the initial condition $U(0)=I$ may be considered. The problem is to show that the solution $U(t)$ can be written in the form $U(t)=\exp\Omega(t)$ with an $\Omega(t)$ obtainable from $A(t)$ by the operations of Lie multiplication and integration processes. A formula is obtained satisfying these requirements [see also corresponding considerations in Feynman, Phys. Rev. (2) **84**, 108–128 (1951); MR **13**, 410]. The convergence questions connected with the formula are then discussed in the second part of the paper, in particular for the finite matrix algebra over the complex field and various sufficient conditions for convergence are obtained.

C. Loewner (Stanford, Calif.).

Masuda, K. Hasse factor systems reduced mod $\mathfrak{k}$. J. Reine Angew. Math. **193**, 161–165 (1954).

Using H. Hasse's work on Galois structures and his own on direct decomposition of such algebras [Hasse, same J. **187**, 14–43 (1949); MR **11**, 576; Masuda, Tôhoku Math. J. (2) **4**, 122–130 (1952); MR **14**, 530], the author obtains a characterization of the decomposition group Z of certain prime ideals P in finite normal extensions K over fields k of Kroneckerian dimension 1 with Galois group G. The prerequisite for the author's work is E. Noether's theorem on normal bases [J. Reine Angew. Math. **167**, 147–152 (1932)] assuring the existence of Galois structures termed p-regular $[p=P\cap k]$ which have p-integral generators A_χ and Hasse factor sets $C_{\chi,\psi}$ (see loc. cit.). For these prime ideals p it is assumed that $p\nmid m$, the g.c.d. of the orders of the elements in the Galois group of K/k; furthermore, all mth roots of unity are to lie in K. Reducing the structural elements A_χ, $C_{\chi,\psi}$ modulo p, a Galois structure over the residue class field of k mod p with the automorphism group G is obtained. Then Z is a minimal subgroup for which there exists at least one p-regular Galois structure having a direct decomposition modulo p. Incidental to the work of the author are proofs of various theorems of Albert, Speiser and the reviewer on the Galois groups of regular extensions of p-adic fields.

O. F. G. Schilling (Chicago, Ill.).

Wolf, Paul. Das Einbettungsproblem galoisscher Algebren. J. Reine Angew. Math. **193**, 166–182 (1954).

This note contains copious remarks and computations related to a conjecture of H. Hasse concerning conditions for the imbedding of an associative and commutative Galois algebra Ω/Ω_0 into a larger Galois algebra K/Ω_0 with prescribed structure of the automorphism group. Relations between the normal factors of K/Ω, Ω/Ω_0 and K/Ω_0 are clarified [see the author's earlier papers, e.g., Abh. Math. Sem. Univ. Hamburg **18**, 179–195 (1952); MR **14**, 531]. Furthermore, a generalization of R. Brauer's imbedding theorem [e.g., in Ann. of Math. (2) **48**, 79–90 (1947); MR **8**, 310] is established. It is to be noted that Hasse's abovementioned conjecture can be validated for general Ω_0 only if the group of K/Ω is cyclic [in this connection see, e.g., D. K. Faddeev Dokl. Akad. Nauk SSSR (N.S.) **94**, 1013–1016 (1954); MR **15**, 938]. *O. F. G. Schilling.*

Orsinger, Heinz. Resultantensysteme und algebraische Relationen. Math. Nachr. **12**, 209–248 (1954).

Let $y_1, \cdots, y_r$ denote complete forms in the variables $x_1, \cdots, x_n$ with indeterminate coefficients $\{a\}$ over a field K. Then a set $R_1, \cdots, R_h$ of polynomials in the a's with integral coefficients is termed a set of resultants for $y_1, \cdots, y_r$ if the simultaneous vanishing of the R_i for a specialization $\{a\}\to\{a^*\}$ and $\{y_i\}\to\{y_i^*\}$ is a necessary and sufficient condition for the existence of a non-trivial zero of the homo-

geneous ideal $(y_1^*, \cdots, y_r^*)$. The author shows that a suitable set of R_i's may be interpreted as coefficients of certain well defined minimal polynomials belonging to extended forms

$$\bar{y}_i = y_i + \sum_{\nu=1}^{r-n} a_{i,n+\nu} x_{n+\nu}^{g_i},$$

where g_i denotes the degree of y_i, $q_{i,n+\nu}$ and $x_{n+\nu}$ being new independent indeterminates; these minimal polynomials are related to the equations which are satisfied by the x's over $K[\{a\}; \{\bar{y}\}]$.

His result generalizes a theorem previously found by O. Perron [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 117–130 (1951); MR **13**, 33] for $r=n$, in which case $h=1$ and $R=R_1$ is the essentially unique basis of the inertial ideal $(R_1, \cdots, R_h) = \{\text{all } T(a) \text{ with } T(a)x_\nu^\tau = 0 \ ((y_1, \cdots, y_r))$ for some ν and $\tau\}$ [see, e.g., B. L. van der Waerden, Moderne Algebra, Bd. 2, 2nd ed., Springer, Berlin, 1940, Chap. 11; MR **2**, 120]. Perron interpreted R ("Perron coefficient") as the coefficient in $Rz^0 + \cdots = 0$ $(G = \prod_{i=1}^{r} g_i)$, the equation satisfied by an x over $K(y_1, \cdots, y_n)$. The author's method consists in reducing the case $r > n$ to that of Perron. He shows (the case $r = n+1$ is general enough to exhibit his approach) that the extended forms $\bar{y}_1, \cdots, \bar{y}_{n+1}$ specialize to forms $y_1^*, \cdots, y_{n+1}^*$ (keeping the new coefficients $a_{i,n+1}$ indeterminate) with a non-trivial zero if and only if the Perron coefficient $\bar{R}$ belonging to $\bar{y}_1, \cdots, \bar{y}_{n+1}$ with respect to $x_1, \cdots, x_{n+1}$ vanishes for the specialization. Furthermore, it is proved that $\bar{R} = R_*(a_{1\rho}, \cdots, a_{n+1,\rho})^d$, $d = (g_1, \cdots, g_{n+1})$, where R_* is irreducible in $K[\{a_{i\rho}\}]$, $1 \leqq \rho \leqq n+1$; and R_* appears in a suitably modified form as the coefficient of the highest term of the irreducible polynomials over $K[\{a\}; \{\bar{y}\}]$ with the zeros x_ρ. Finally, it is shown that the coefficients attached to the power products of the $a_{\nu,n+1}$, $1 \leqq \nu \leqq n+1$ in R_* constitute a system of resultants for $y_1, \cdots, y_{n+1}$. Moreover, these coefficients are determined (as conjectured by Perron) as those of minimal equation

$$R_*(-y_1, \cdots, -y_{n+1}) = 0$$

of y_ρ over

$$K[\{a\}; y_1, \cdots, y_{\rho-1}, y_{\rho+1}, \cdots, y_{n+1}];$$

this equation of weight G/d in the x's is of minimal weight and essentially uniquely determined.

O. F. G. Schilling (Chicago, Ill.).

Flanders, Harley. A remark on Hilbert's Nullstellensatz. J. Math. Soc. Japan **6**, 160–161 (1954).

A lemma of Zariski states that if k and $K = k[x_1, x_2, \cdots, x_n]$ are fields, then the quantities x_i are algebraic over k. The author offers the following proof, based on the notion of integral closure of a ring. Assume $x_1, x_2, \cdots, x_r$ to be a transcendence base for K/k, put $o = k[x_1, x_2, \cdots, x_r]$, and select a non-zero element $f \in o$ such that fx_j is integral over o for each $j > r$. Then for each $\alpha \in K = k[x_1, x_2, \cdots, x_n]$ it is clear that $f^\nu \alpha$ is integral over o for suitable ν. Taking $\alpha = 1/g$, $g \in o$, it follows that g divides some power of f; but if $r > 0$ this is impossible, e.g. for $g = 1 + x_1 f$. (Flander's belief that a certain norm formation is the crux of his proof is incorrect.)

J. Tate (Cambridge, Mass.).

Jaffard, Paul. La notion de valuation. Enseignement Math. **40** (1951–1954), 5–26 (1955).

An expository article, sketching the important concepts in valuation theory and some applications to divisibility questions.

D. Zelinsky (Evanston, Ill.).

Theory of Groups

***Alexandroff, P. S. Einführung in die Gruppentheorie.** Deutscher Verlag der Wissenschaften, Berlin, 1954. vii+120 pp.

This little book is a translation of the author's Vvedenie v teoriyu grupp [Učpedgiz, Moscow, 1951]. It is introductory in the true sense of the word. The beginner is led into the study of basic concepts of elementary group theory via a sequence of short, easy and above all clear paragraphs. In keeping with the purpose of the book, the reader is never taken beyond the fundamentals. Ch. I (The group concept) covers the axioms and contains some examples, typical of which is the Klein four group. Ch. II (Permutation groups) ends with the determination of the order of the alternating group of degree n. Ch. III (Some general remarks on groups, The concept of isomorphism) contains the Cayley representation theorem. Ch. IV (Cyclic subgroups of a given group) emphasizes the geometric interpretation of the groups of its heading. In Ch. V (Simple groups of motions) the group of rotations of the icosahedron is the last topic. The level of the book is indicated here by the fact that the author determines the order of this group but then states without proof its isomorphism with the alternating group of degree 5. Ch. VI (Invariant subgroups) contains as its deepest result the theorem that the group of parallel displacements of the plane is invariant in the group of motions of the plane. Ch. VII (Homomorphic mappings) features the construction of all finite cyclic groups as homomorphic images of the group of integers. Chapter VIII (Decomposition of groups into classes with respect to a given subgroup, Groups of cosets) ends with a summary of facts concerning the kernel of a homomorphism and the representation of a homomorphic image as a group of cosets. The book closes with a short appendix, Elementary concepts of set theory. There are about 17 line drawings and about 10 group or coset tables.

F. Haimo (St. Louis, Mo.).

***Tvermoes, Helge. Om en Generalisation af Gruppebegrebet. [On a generalization of the notion of group.]** Thesis, University of Copenhagen, 1952. xii+107 pp.

A n-group ['n-Gruppe' in Danish] is an algebra with one operation κ which is $(n+1)$-ary, associative, and such that

$$\kappa a_0 a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n = a$$

is solvable for every i from 0 to n and every $a, a_0 \cdots a_n$. In fact, it is the same as the $(n+1)$-Gruppe of W. Dörnte [Math. Z. **29**, 1–19 (1928)] or the $(n+1)$-adic group of E. L. Post [Trans. Amer. Math. Soc. **48**, 208–350 (1940); MR **2**, 128]. Tvermoes seems not to have seen Post's paper, for it is not mentioned in the Bibliography, although it contains most of Tvermoes' results, including the basic theorem that a n-group can always be embedded in a group, and is in fact a coset of a normal subgroup ['Hovedgruppe' (literally: 'head-group') in Danish] whose factor-group is cyclic.

The first section of the thesis is concerned with known results from ordinary group theory used in the later sections.

Section II contains the definitions of n-group and n-semigroup, examples thereof, the basic theorem for n-groups, and some direct consequences thereof. An n-group is called "proper" if its operation is not the continued product of an r-ary operation on the same set of elements for any proper factor r of n. Criteria (equivalent to Post's) are given for a n-group to be proper.

Section III is concerned with the determination of proper 'Hovedgruppe', presumably as a step in determining all proper n-groups. *H. A. Thurston* (Bristol).

Lyapin, E. S. Semigroups in all of whose representations the operators have fixed points. II. Mat. Sb. N.S. **36(78)**, 111–124 (1955). (Russian)

This is the second part of an earlier article by the same author under the same title [Mat. Sb. N.S. **34(76)**, 289–306 (1954); MR **15**, 850]. In the first part, the author exhibited a "basic class" P_0 of the class P of all semigroups described in the title. In the present paper, the author shows how to construct all possible basic classes of P. If a semigroup $\mathfrak{A}$ is represented in two ways as right-annihilating products,

$$\mathfrak{A}=\mathfrak{B}_1\underset{\phi_1}{\times}\mathfrak{M}_1=\mathfrak{B}_2\underset{\phi_2}{\times}\mathfrak{M}_2,$$

where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both cyclic, then $\mathfrak{M}_1=\mathfrak{M}_2$, $\mathfrak{B}_1\cong\mathfrak{B}_2$, and for $M_1\in\mathfrak{M}_1$, $M_2\in\mathfrak{M}_2$ we have $\phi_1M_1=\phi_1M_2$ if and only if $\phi_2M_1=\phi_2M_2$. The uniquely determined cyclic subsemigroup $\mathfrak{M}_1$ of $\mathfrak{A}$ is called the nucleus of $\mathfrak{A}$. Two semigroups $\mathfrak{A}_1$ and $\mathfrak{A}_2$ having nuclei $\mathfrak{M}_1$ and $\mathfrak{M}_2$ resp. are called conjugate if there exists a subsemigroup of $\mathfrak{A}_1$ containing $\mathfrak{M}_1$ and isomorphic with $\mathfrak{A}_2$, and vice-versa. An explicit description is given of every semigroup which is conjugate with some member of P_0. Two classes Q_1 and Q_2 of semigroups containing nuclei are called conjugate if each member of Q_1 is conjugate with one and, to within isomorphism, only one member of Q_2, and vice-versa. The main theorem then is: a class of semigroups is a basic class of P if and only if it is conjugate with P_0. *A. H. Clifford* (Baltimore, Md.).

Clifford, A. H. Note on Hahn's theorem on ordered abelian groups. Proc. Amer. Math. Soc. **5**, 860–863 (1954).

Für den Satz, dass jeder totalgeordnete reelle Vektorraum in einem lexikalisch geordneten reellen Funktionenraum eingebettet werden kann, gibt es einen kurzen Beweis von Hausner und Wendel [dieselben Proc. **3**, 977–982 (1952); MR **14**, 566]. Schon Hahn [Akad. Wiss. Wien, S.-B. IIa. **116**, 601–655 (1907)] hat aber diese Einbettbarkeit allgemeiner für jede totalgeordnete abelsche Gruppe bewiesen. Verf. zeigt nun, wie der Hausner-Wendelsche Beweis modifiziert werden kann, um den Hahnschen Satz zu erhalten. Es wird dazu jede totalgeordnete abelsche Gruppe zunächst in einen totalgeordneten rationalen Vektorraum eingebettet. Der Übergang von "reell" zu "rational" erfordert nur an einer Stelle eine Modifikation.

P. Lorenzen (Bonn).

Ree, Rimhak. On ordered, finitely generated, solvable groups. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 39–42 (1954).

The author proves that in an ordered group with maximal condition for subgroups the lower central series contracts (possible transfinitely) to the 1-element. If in addition the group is assumed to be solvable, then it is nilpotent of finite class. Conversely, a finitely generated, nilpotent, ordered group satisfies the maximal condition for subgroups. [Reviewer's remark: this is true even without the assumption that the group is ordered.] Finally, the author shows by a simple example that a finitely generated, ordered, solvable group need not be nilpotent. *K. A. Hirsch* (London).

Lyndon, R. C. On Burnside's problem. II. Trans. Amer. Math. Soc. **78**, 329–332 (1955).

This is a continuation of part I [same Trans. **77**, 202–215 (1954); MR **16**, 218] by the same author. It is proved here that, in the notation of the previous review,

(A) if $q=2$ and $p\geqq 5$, then $\delta(p+3)=6p-4$;
(B) if $q=2$ and $p=5$, then $K(q)$ is 4, 2 or 0.

The proofs are given in outline only. *P. Hall.*

Taunt, D. R. Remarks on the isomorphism problem in theories of construction of finite groups. Proc. Cambridge Philos. Soc. **51**, 16–24 (1955).

The paper gives a method of constructing A-groups, that is, solvable groups whose Sylow subgroups are all abelian, and in particular yields a method for constructing all solvable groups of cube-free order. The approach uses the extension of a certain automorphism from a quotient group to the group itself, and so the author also studies a certain class of p-groups where isomorphisms between quotient groups of the original group are induced by automorphisms of that group. A group G is said to have property E if whenever M and N are normal subgroups of G then any isomorphism between G/M and G/N can be extended to an automorphism of G. It has property E' if every automorphism of G/N can be lifted to one of G. He then proves, for instance, that if a p-group has property E' then its Frattini subgroup is its unique maximal proper characteristic subgroup, and that any Abelian group of cube-free order has property E. Given two groups M and N and a homomorphism φ of M into the automorphism group of N, he defines $G_p[M, N, \varphi]$ to be all pairs (m, n), where

$$(m_1, n_1)(m_2, n_2)=(m_1m_2, n_1{}^{\varphi(m_2)}n_2).$$

Given two groups G_1 and G_2 where $G_i=G_p[M, N, \varphi_i]$, he proves that a sufficient condition that G_1 and G_2 be isomorphic is that there exist automorphisms χ, ω of M and N respectively so that $\varphi_2(m^\chi)=\omega^{-1}\varphi_1(m)\omega$ for all $m\in M$. He uses these results to construct all solvable groups of cube-free order. *I. N. Herstein* (Philadelphia, Pa.).

Taunt, D. R. Finite groups having unique proper characteristic subgroups. I. Proc. Cambridge Philos. Soc. **51**, 25–36 (1955).

A finite group is said to be a UCS group if it has exactly one proper characteristic subgroup. The author is concerned (amongst other things) with the problem of determining when the direct product of UCS groups is again a UCS group; he intensively studies UCS groups whose orders involve only two primes. The following two theorems are samples of his results. 1. If G is a UCS group and N its characteristic subgroup and if a) no element of G other than the identity is left fixed by all the automorphisms of G, and b) no element of G/N other than the identity is left fixed by all the automorphisms of G/N induced by automorphisms of G, then the direct product of G by itself a finite number of times is a UCS group. 2. Let G be a UCS group of order p^rq^s, p, q primes, with characteristic subgroup of order p^r; then the direct product of G with itself any finite number of times is UCS except possibly if $s=1$ and q divides $p-1$.

I. N. Herstein (Philadelphia, Pa.).

Zitarosa, Antonio. Sulla relazione tra il tipo ed il rango di un gruppo finito. Boll. Un. Mat. Ital. (3) **9**, 164–167 (1954).

G. Zappa [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **6**, 361–365, 441–445 (1927); Rend. Sem. Mat. Univ. Padova **13**, 36–40 (1942); MR **8**, 311] proved that if τ is the type and ρ the rank of a group, then

(I) $\tau+2\geqq 2^{\rho+1}-1$.

He also conjectured that

(II) $$r+2 \geqq 3(2^p-1)$$

and that (III) for each p there exist groups for which $r+2=3(2^p-1)$. The present paper sharpens (I) to

$$r+2 \geqq 2^{p+1}+p-1$$

if $p \geqq 2$ and proves (III). *H. A. Thurston* (Bristol).

Fryer, K. D. A class of permutation groups of prime degree. Canad. J. Math. **7**, 24–34 (1955).

In this thesis the author studies permutation groups $\{A, B, C\}$ whose generators A, B, C are of orders $p=2q+1$, q, and $r=(q-1)/s$, respectively. Those groups that contain odd permutations (Case II) are the symmetric groups S_p. Those besides the alternating groups that contain only even permutations (Case I) are all simple, and for $p \leqq 59$, there are only four such groups, namely $LF(2, 7)$, $LF(2, 11)$, and the Mathieu groups M_{11} and M_{23}. First the author combines the representation $A: x \to x+1$, $B \to g^2x$ (mod p) of the metacyclic group $\{A, B\}$ with a similar representation on the exponents of x and g for the metacyclic group $\{B, C\}$. The generators B and C each fix 0, and C either fixes 1 and g^β if C is even, or interchanges them if it is odd. In the latter case $\{ABC\}$ is the symmetric group. When C is even, r even and $\beta=q$, $\{A, B, C\}$ is the alternating group. Only the four other simple groups mentioned above have appeared although it was hoped that new simple groups might have been discovered in this manner. *J. S. Frame.*

Beaumont, R. A., and Peterson, R. P. Set-transitive permutation groups. Canad. J. Math. **7**, 35–42 (1955).

The alternating and symmetric groups A_n and S_n (except A_2) are set-transitive in the sense that they are s-set transitive for all s. All set-transitive groups are shown to be transitive and primitive ($s>1$), and it is further shown that apart from A_n and S_n, the only four types of set-transitive groups are the metacyclic group of degree 5 and order 20, a group of degree 6 and order 120 isomorphic with S_5, and two groups of degree 9: the group $LF(2, 8)$ of order 504 and its group of automorphisms of order 1512. Although the result has been attributed to Chevalley, the authors have not seen it in print. Important steps in their development are (1) the order of a set-transitive subgroup of S_n is divisible by the least common multiple of the binomial coefficients $\binom{n}{i}$; and (2) if a prime p exists between $(n+1)/2$ and $2n/3$ then a subgroup of S_n cannot be $[n/2]$ set-transitive unless it contains A_n. This eliminates all $n>16$ except $n=25$. Other theorems exclude all n but 5, 6, 9, and Cole's enumeration of transitive subgroups is used for the final reduction to four groups. *J. S. Frame* (East Lansing, Mich.).

Zappa, Guido. Sopra un'estensione di Wielandt del teorema di Sylow. Boll. Un. Mat. Ital. (3) **9**, 349–353 (1954).

Wielandt has proved recently [Math. Z. **60**, 407–408 (1954); MR **16**, 331] the following extension of Sylow's Theorem: If a group G of order g contains a nilpotent subgroup H whose order H is prime to its index g/h, and if M is a subgroup of G whose order m is a divisor of h, then M is contained in a subgroup conjugate to H. It is an open question whether it is sufficient to assume that H is supersoluble. In the present note the author proves a corresponding theorem for dispersible groups. According to Ore [Duke Math. J. **5**, 431–460 (1939)] a group of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ ($p_1>p_2>\cdots>p_k$, p_i primes) is called dispersible if for every i it contains a normal subgroup of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_i^{\alpha_i}$. Theorem. Let M and H be two dispersible subgroups of a finite group G whose orders are prime to their indices in G, and let the order of M divide the order of H. Then M is contained in a subgroup of G conjugate to H.

K. A. Hirsch (London).

Suprunenko, D. On nilpotent transitive subgroups of the symmetric group. Dokl. Akad. Nauk SSSR (N.S.) **99**, 23–25 (1954). (Russian)

The principal result of this note is the theorem that all maximal transitive nilpotent subgroups of a finite symmetric group are conjugate. More precisely: let S_n be the symmetric group of degree n, $n=n_1\cdots n_k$, $n_i=p_i^{\alpha_i}$ (p_i primes), N_{p^α} a p-Sylow subgroup of S_{p^α}. Then S_n contains a transitive nilpotent subgroup $N_{(n)}$ which is isomorphic to the direct product $N_{n_1}\times\cdots\times N_{n_k}$; every transitive nilpotent subgroup of S_n is conjugate to one of the subgroups of $N_{(n)}$ and, in particular, all the maximal transitive nilpotent subgroups of S_n are isomorphic to $N_{(n)}$ conjugate. Their order is, of course, $p_1^{\beta_1}\cdots p_k^{\beta_k}$, where $\beta_i=(p_i^{\alpha_i}-1)/(p-1)$.

K. A. Hirsch (London).

Suprunenko, D. A. On irreducible nilpotent matrix groups. Mat. Sb. N.S. **35(77)**, 501–512 (1954). (Russian)

For the purposes of this review the term group shall mean "group of matrices over an algebraically closed field P of characteristic p" (the case $p=0$ is not excluded). In a previous paper [Mat. Sb. N.S. **31(73)**, 353–358 (1952); MR **14**, 447] the author has studied irreducible nilpotent groups of prime degree. The results obtained there are extended in the present paper to groups of squarefree degree. The principal theorems are as follows. If n is divisible by p, then $\mathrm{GL}(n, P)$ has no irreducible nilpotent subgroups. But otherwise—in particular, if P is of characteristic zero—$\mathrm{GL}(n, P)$ contains irreducible nilpotent subgroups of arbitrary class l. If n is not divisible by p, then the maximal irreducible nilpotent subgroups of $\mathrm{GL}(n, P)$ of class 2 fall into as many classes of conjugate subgroups as there are partitions of n. If n is squarefree and not divisible by p, then $\mathrm{GL}(n, P)$ contains a countable chain of subgroups $\Gamma_n^2 \subset \Gamma_n^3 \subset \cdots \subset \Gamma_n^l \subset \cdots$ with the following properties: (i) Γ_n^l is a nilpotent irreducible group of class l; (ii) every irreducible nilpotent subgroup of $\mathrm{GL}(n, P)$ of class l is conjugate to Γ_n^l or to one of its subgroups; (iii) the factor groups Z_{i+1}/Z_i for $i=1, 2, \cdots, l-2$ are cyclic of order n, while Γ_n^l/Z_{l-1} is the direct product of two cyclic groups of order n so that the index of the center of Γ_n^l is n^1. The fact that P is algebraically closed is essential here: over the field of real numbers there are no irreducible nilpotent groups of odd degree. *K. A. Hirsch* (London).

Plotkin, B. I. On the theory of solvable groups with finiteness conditions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 417–420 (1955). (Russian)

Let $R(G)$, the radical of a group G, be that (characteristic and locally nilpotent) subgroup of G generated by all the locally nilpotent subgroups of G. Form an ascending series of radicals in the obvious fashion (constructing the inverse image of the radical in the quotient by the preceding radical); and if the (possibly transfinite) series of radicals terminates with G, then G is said to be a radical group. Theorem 1 states that a radical group is finitely generated if its radical is so generated. The author calls a group special if it is the extension by a finite group of a direct sum of a finite number of groups of types p^∞. Let G be a radical group

with a special radical. If G is also periodic (Th. 2) or has the minimum condition on normal subgroups (Th. 3), then G is special. If (Th. 4) G is a radical group of finite special rank with the minimum condition for normal subgroups, then G has the minimum condition for subgroups. *F. Haimo.*

Haimo, Franklin. Normal automorphisms and their fixed points. Trans. Amer. Math. Soc. **78**, 150–167 (1955).

Let Z_n be the nth member of the ascending central series of an arbitrary group G. An automorphism α of G is called n-normal automorphism if it has the property $\alpha(x) \equiv x \bmod Z_n$ for every $x \in G$. If T_n denotes the set of all n-normal automorphisms of G, then in particular T_1 is the centralizer of the group of inner automorphisms of G (in the group A of automorphisms of G). Let B_1 be the set of fixed points held in common by the mappings from T_1. The factor group G/B_1 is abelian and the elements of T_1 which induce either the identity or the involution automorphism on G/B_1 form a subgroup W of T_1. After investigation of the ascending central series of W, the author defines the B-series as follows: B_1 as before; moreover, if B_j is already defined, then let B_{j+1} be the union in G of all cosets relative to B_j being fixed for all automorphisms of G/B_j. Each B_j is a normal subgroup of G, the factor groups G/B_j are abelian, and the B-series ascends monotonically in its index. The author obtains a number of properties of B-series of G. It is shown that the elements of G contained in B_n are fixed points for 2^{n-1}-th powers of every 1-normal automorphism. For n-normal automorphisms the author proves that the common fixed points lie in the centralizer of Z_n in G. At the end of the paper interesting examples are discussed.

A. Kertész (Debrecen).

Higman, D. G. Modules with a group of operators. Duke Math. J. **21**, 369–376 (1954).

G sei eine (multiplikative) Gruppe, M ein G-Modul, d.h. eine (additive) Abelsche Gruppe mit G als Linksoperatoren-Gruppe; ist U eine Untergruppe von G, so bezeichnet M_U den U-Modul, den man durch Beschränkung der Operatoren auf U erhält. Für endlichen Index $n = G{:}\,U$ untersucht der Verf., ob (1) für jeden G-Modul $H \supset M$, für welchen M_U direkter Summand in H_U ist, auch M direkter Summand in H ist. Notwendig und hinreichend hiefür ist (2) die Existenz eines Endomorphismus α von M_U, derart dass $\sum_{i=1}^{n} x_i \alpha x_i^{-1}$ die Identität von M ist; $x_1, \cdots, x_n$ bezeichnet ein beliebiges Repräsentantensystem der Linksrestklassen von G modulo U. Aequivalent damit ist: (3) Der G-Modul $(M_U)^G$ enthält einen zu M isomorphen Teilmodul. Hiebei bedeutet N^G, für einen U-Modul N, den "induzierten G-Modul" im Sinne der Darstellungstheorie; es existiert stets eine natürliche isomorphe Einbettung β von M in $(M_U)^G$, sowie eine "Verlagerung" β' von $(M_U)^G$ auf M [vgl. B. Eckmann, Proc. Nat. Acad. Sci. U. S. A. **39**, 35–42 (1953); Ann. of Math. (2) **58**, 481–493 (1953); MR **15**, 459, 397]. Aequivalent zu (1)–(3) ist ferner jede der folgenden Bedingungen: (4) M ist direkter Summand in $(M_U)^G$, oder: (5) der Kern von β' ist direkter Summand in $(M_U)^G$. Handelt es sich durchwegs um Moduln mit einem weitern Bereich Ω von Rechtsoperatoren, und sind die Teilmoduln, Homomorphismen usw. stets zulässig bezüglich Ω, so bleiben alle Sätze unverändert gültig. In den genannten Resultaten sind Sätze von Gaschütz [Math. Z. **56**, 376–387 (1952); MR **14**, 533] sowie von Eckmann [loc. cit.] als Spezialfälle enthalten.

Anwendung: G sei endlich, $\Omega = F$ ein Körper der Charakteristik p, und U enthalte eine p-Sylowgruppe von G. Dann ist das Kriterium (2) erfüllt mit $\alpha =$ Division durch n; wenn also eine Darstellung von G als Darstellung von U betrachtet zerfällt, und zwar in zwei Bestandteile, von denen einer gegenüber G zulässig ist, so zerfällt sie auch bezüglich G. Eine weitere Anwendung betrifft die Aequivalenz von Darstellungen. *B. Eckmann* (Zürich).

Higman, D. G. Indecomposable representations at characteristic p. Duke Math. J. **21**, 377–381 (1954).

Im Rahmen der Betrachtungen des Verf. in der vorstehend referierten Arbeit und von Eckmann [loc. cit.] werden Darstellungen einer endlichen Gruppe G über einem Körper der Charakteristik p untersucht. Es wird gezeigt (Bezeichnungen wie oben): (a) Enthält die Untergruppe U von G eine p-Sylowgruppe von G, so ist jeder unzerlegbare Modul M äquivalent einem direkten Summanden von $(M_U')^G$, für einen gewissen unzerlegbaren direkten Summanden M_U' von M_U. (b) Ist U eine beliebige Untergruppe von G, so ist jeder unzerlegbare Darstellungsmodul m von U äquivalent einem direkten Summanden von $(m^G)_U'$, für einen gewissen unzerlegbaren direkten Summanden $(m^G)'$ von m^G. Hieraus ergibt sich: (c) Eine Gruppe G mit lauter zyklischen p-Sylowgruppen hat bei Charakteristik p nur endlich viele Klassen unzerlegbarer Darstellungen, eine Gruppe mit nicht-zyklischen p-Sylowgruppen dagegen unzerlegbare Darstellungen beliebig hohen Grades. Zum Beweis des zweiten Teiles von (c) konstruiert der Verf. unzerlegbare Darstellungen beliebig hohen Grades für eine Gruppe vom Typus (p, p). *B. Eckmann* (Zürich).

Osima, Masaru. On the representations of the generalized symmetric group. Math. J. Okayama Univ. **4**, 39–56 (1954).

All permutations of the mn symbols which commute with

$$(1_1 2_1 \cdots m_1)(1_2 2_2 \cdots m_2) \cdots (1_n 2_n \cdots m_n)$$

constitute a group $S(n, m)$ of order $n!m^n$ which the author calls the generalized symmetric group. Clearly,

$$Q_i = (1_i 2_i \cdots m_i) \in S(n, m)$$

for all i, and $\mathfrak{Q} = \{Q_i\}$ is an invariant abelian subgroup of $S(n, m)$ of order m^n; every element of $S(n, m)$ is of the form W^*Q, where $W^* \in S_n^* \sim S_n$ and $Q \in \mathfrak{Q}$, so that $S(n, m)/\mathfrak{Q} \sim S_n$. After reviewing the pertinent general representation theory, the author proves that the irreducible representations $(\alpha)^*$ of $S(n, m)$ are in 1-1 correspondence with the skew representations

$$[\alpha]_m^* = [\alpha_0] \cdot [\alpha_1] \cdot \cdots \cdot [\alpha_{m-1}]$$

[cf. Robinson, Amer. J. Math. **70**, 277–294 (1948); MR **10**, 678], where each disjoint Young diagram $[\alpha_i]$ contains n_i nodes and $\sum n_i = n$ $(0 \leqq n_i \leqq n)$. The degree of $(\alpha)^*$ is equal to the degree of $[\alpha]_m^*$. Just as in the case of S_n we have a method for calculating the characters of $S(n, m)$ which generalizes the Murnaghan-Nakayama recursion formula and is derived in the same manner [cf. Robinson, ibid. **69**, 286–298 (1947); MR **8**, 563].

The purpose of introducing and studying $S(n, m)$ is to obtain information concerning the modular representation theory of S_n. If b p-hooks can be removed from $[\alpha]$, then $S(b, p)$ is the significant group, and by utilising properties of characters already obtained [Osima, Canad. J. Math. **6**, 511–521 (1954); MR **16**, 566] the author is able to prove the following result. Let $(d_{\alpha\lambda})$, $(\bar{d}_{\alpha'\lambda'})$ be the decomposition matrices of p-blocks B and $\bar{B}$ of the same weight b respectively, and let $[\alpha]$ and $[\alpha']$ have the same star diagrams,

then

$$(\sigma_{\alpha'}\bar{d}_{\alpha'\lambda'}) = (\sigma_\alpha d_{\alpha\lambda})(w_{\lambda\lambda'})$$

where the $w_{\lambda\lambda'}$ are rational integers with $|w_{\lambda\lambda'}| = \pm 1$, and $\sigma_\alpha = \pm 1$. An immediate consequence of this result is that two matrices of Cartan invariants corresponding to p-blocks of the same weight have the same elementary divisors.

G. de B. Robinson (Toronto, Ont.).

Ramanathan, K. G. A note on symplectic complements. J. Indian Math. Soc. (N.S.) **18**, 115–125 (1954).

The symplectic group $\mathrm{Sp}(m)$ is the set of all real matrices M such that $M'JM = J$, where $J = J_m = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and E is the unit matrix of order m (M' is the transposed of M). The modular group of degree m is the subgroup of all integral symplectic matrices. A symmetric matrix S of $2m$ rows is called abelian if $SJS = \lambda J$, where $\lambda \neq 0$. If S and T are two rational abelian matrices of $2m$ and $2n$ rows respectively (where $m \geqq n$), then $T = X'SX$ is said to be a symplectic representation of T by S, if $X'JX = \lambda J$, where $\lambda \neq 0$. It is said to be primitive if $\lambda = 1$. A matrix Y of $2m$ rows and $2m-2n$ columns is said to be a complement of a primitive X if $Y'J_mY = J_{m-n}$ and $(CFDG)$ is a modular matrix, if $X = (CD)$, $Y = (FG)$ where C, D have $2m$ rows and n columns and F, G have $2m$ rows and $m-n$ columns. The author proves that complements of a primitive X exist and obtains a parametric representation for all complements of a primitive X.

H. D. Kloosterman (Leiden).

Tachikawa, Hiroyuki. A remark on generalized characters of groups. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **4**, 332–334 (1954).

Using methods developed by P. Roquette [J. Reine Angew. Math. **190**, 148–168 (1952); MR **14**, 844], the following theorem of R. Brauer is proved: "A complex-valued function φ on a finite group G is a generalized character of G (i.e. an integral combination of absolutely irreducible characters) if and only if (i) φ is a conjugate class function, and (ii) for every elementary subgroup the restriction of φ to the subgroup is a generalized character of the subgroup" [see R. Brauer, Ann. of Math. (2) **57**, 357–377 (1953); MR **14**, 844].

O. F. G. Schilling (Chicago, Ill.).

Gel'fand, I. M., and Graev, M. I. Traces of unitary representations of the real unimodular group. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1037–1040 (1955). (Russian)

The authors give an explicit formula for the 'trace' in the case of the principal series of irreducible continuous unitary representations of the real unimodular group G in n dimensions [for the definition of this series, see the authors' paper Izv. Akad. Nauk. Ser. Mat. **17**, 189–248 (1953); MR **15**, 199]. This shows in particular that the trace does in fact exist as a function on the group. That is, if x is a square-integrable function on G that vanishes outside a compact set, and if T is one of the stated representations, then: (1) $T_x = \int_G T(g)x(g)dg$ is an operator of Hilbert-Schmidt class, (2) the trace of T_x (in the usual operator-theoretic sense) has the form $\int_G x(g)\pi(g)dg$, for some fixed function π, called the trace of the representation.

In the case of complex semi-simple Lie groups, analogous results had been obtained earlier by Gel'fand and Naĭmark [Trudy Mat. Inst. Steklov. **36** (1950); MR **13**, 722]. The real case is however significantly more complicated. In the case $n=2$ the situation was known through the work of Bargmann and Harish-Chandra [see the latter's formula for the trace in this case in Proc. Nat. Acad. Sci. U. S. A. **38**, 337–342 (1952); MR **13**, 820].

I. E. Segal.

Mostow, G. D. On covariant fiberings of Klein spaces. Amer. J. Math. **77**, 247–278 (1955).

By a Klein space, the author means the factor space G/C of a Lie group G by a closed subgroup C. In the following, C is assumed to be connected. A G-covariant fibering of $H = G/C$ is a decomposition of the differential manifold H into Euclidean fibers such that the fibers are permuted transitively by some maximal compact subgroup M of G and that the subgroup of M which keeps some fiber F invariant is equivalent to a linear group on F.

The main results of the paper are as follows. 1) An aspherical Klein space is homeomorphic to Euclidean space. 2) A Klein space G/C can be retracted by a strong retraction to a subset homeomorphic to the compact space $M/M \cap C$, where M denotes a maximal compact subgroup of G containing a maximal compact subgroup of C. 3) Let R be the radical of G. If the group CR/R is invariant under a so-called star automorphism of the semi-simple Lie group G/R, then G/C admits a G-covariant fibering. The condition is satisfied if C is either semi-simple or solvable. 4) Any two G-covariant fiberings of a Klein space are equivalent in the sense that there exists a one-one bundle map of one onto the other.

The proof is based upon theorems on decomposing connected Lie groups into products of subgroups and subsets homeomorphic to Euclidean spaces. For example: Let G be a connected Lie group. Then it contains a maximal compact subgroup M and a Euclidean subspace U such that $G = MU$ topologically, and $mUm^{-1} = U$ for all m in M.

K. Iwasawa (Cambridge, Mass.).

Dobrescu, Andrei. Sur les groupes de Lie à trois paramètres. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **5**, 75–81 (1953). (Romanian. Russian and French summaries)

The real, 3-dimensional Lie algebras have been classified by L. Bianchi [Lezioni sulla teoria dei gruppi continui finiti di trasformazioni, Spoerri, Pisa, 1918], G. Vrănceanu [Leçons de géométrie différentielle, v. 1, Bucarest, 1947; MR **9**, 532], and H. C. Lee [J. Math. Pures Appl. (9) **26**, 251–267 (1948); MR **9**, 567], by means of different methods. The author establishes the equivalence between the three classifications.

J. L. Tits (Brussels).

Matsushita, Shin-ichi. Sur le théorème de Plancherel. Proc. Japan Acad. **30**, 557–561 (1954).

Let G be a locally compact group and let V_0 be the locally compact space obtained by removing the 0 element from the weak closure of the set of all elementary continuous positive definite functions on G. For each $\varphi \in V_0$ let $(f, g)_\varphi = \int_G \overline{\phi(x)}(g^* f)(x)\sqrt{\rho(x)}dx$ for all continuous f and g on G with compact support. Here dx denotes left invariant Haar measure and ρ is the factor compensating for possible lack of right invariance of this measure. In the case of a discrete G the author's Plancherel theorem may be stated as follows: There exists a Radon measure μ on V_0 such that $\int_G f(x)\overline{g(x)}dx = \int (f, g)_\phi d\mu(\phi)$ for all continuous f and g with compact support. The proof is an immediate corollary of Godement's [Trans. Amer. Math. Soc. **63**, 1–84 (1948), Th. 12; MR **9**, 327] generalized Bochner theorem; the measure μ being any one corresponding to the positive definite function which is one at the identity and zero elsewhere. In the general case the author does not obtain a measure in V_0 but a family μ_λ of measures produced by

applying the generalized Bochner theorem to the members of a family of positive definite functions whose supports converge to the identity. His theorem asserts that $\int_G f(x)\overline{g(x)}dx=\lim_\lambda \int (f,g)_\phi d\mu_\lambda(\phi)$. When G is abelian further argument leads to the usual Plancherel theorem for Abelian groups. The Plancherel theorem formulated here is rather different from the usual one which produces an actual measure in the dual object which is moreover essentially unique and corresponds to the essentially unique decomposition of the regular representation of G into factor representations. The Plancherel theorem of the paper under review corresponds to an arbitary decomposition of the regular representation into irreducible parts.

G. W. Mackey (Cambridge, Mass.).

Wendel, J. G. Haar measure and the semigroup of measures on a compact group. Proc. Amer. Math. Soc. **5**, 923–929 (1954).

The theorem that a compact semigroup which is not a group must possess non-trivial idempotents is applied by the author to the convolution semigroup S of probability measures on a compact group G. It is shown that the idempotents of S are exactly the Haar measures of compact subgroups of G, from which follows a new proof of the existence of Haar measure on G. *L. H. Loomis.*

Reiter, Hans. Über L^1-Räume auf Gruppen. II. Monatsh. Math. **58**, 172–180 (1954).

[For part I see Monatsh. Math. **58**, 73–76 (1954); MR **16**, 220.] Let G be a locally compact group and let g be a closed Abelian subgroup such that the left quotient space G/g has an invariant measure. If $f\in L^1(G)$, then $\int_g f(xt)dt$ is constant on the cosets of g and so defines a function f' on the quotient space G/g, and the measure on G/g can be normalized so that $\int_G f(x)dx=\int_{G/g} f'(x')dx'$. Now let f be a fixed function of $L^1(G)$ and let N be the closed subspace of $L^1(G)$ spanned by the differences $f(x)-f(xt)$, $t\in g$. The theorem proved by the author can then be stated as follows: The norm of f' in $L^1(G/g)$ is equal to the distance from f to N in $L^1(G)$. *L. H. Loomis* (Cambridge, Mass.).

Hermann, Robert. Sur les espaces homogènes compacts de caractéristique positive. C. R. Acad. Sci. Paris **240**, 1303–1305 (1955).

Many theorems are given on compact homogeneous spaces of positive Euler-Poincaré characteristic. We quote the following typical ones: I) Let G be a simple group and L a maximal subgroup of the same rank. If the center of L is non-discrete, L is a symmetric subgroup. If $\pi_2(G/L)=0$ and if L is of the same structure as a symmetric subgroup of G, then L is itself a symmetric subgroup. In particular, if G is a classical group, L is symmetric. II) A compact homogeneous space V is called an M-space, if any transitive and effective group acting on V has the property that its isotropic subgroup is maximal. Then each of the following conditions implies that V is an M-space. a) The cohomology ring $H^*(V,R)$ of V with real coefficients is generated by an element of $H^p(V,R)$, $p\geqq 4$. b) $H^i(V,R)=0$ for $i\leqq 4$ and V is irreducible in the sense that, with the exception of F_4/Spin (8), it is not the product of manifolds of lower dimensions. c) Dim $V=4n$ and $H^*(V,R)$ is generated by an element of $H^2(V,R)$. III) The spaces G/L, quotients of a simply-connected simple group by a simply-connected non-symmetric maximal subgroup, admit an invariant almost complex structure. Only brief arguments leading to proofs are indicated. *S. Chern* (Chicago, Ill.).

Wallace, A. D. The structure of topological semigroups. Bull. Amer. Math. Soc. **61**, 95–112 (1955).

A motivated summary of the bulk of the existing literature on topological semigroups (mobs) is presented. For the most part, compactness assumptions are made. Although few complete proofs are given, the exposition is otherwise self-contained. A number of the currently outstanding problems in this area are stated. [In Cor. 1, p. 107, and Cor. 2, p. 108, read "$H^n(N,G)\neq 0$" for "$H^n(N,G)=0$."]

M. Henriksen (Lafayette, Ind.).

NUMBER THEORY

Sexton, Charles R. Computo del numero delle coppie di numeri primi gemelli comprese tra 100.000 e 1.100.000, distinte secondo le cifre terminali. Boll. Un. Mat. Ital. (3) **10**, 99–101 (1955).

Duncan, D. G. A generalization of the Euler-Fermat theorem. Amer. Math. Monthly **62**, 241 (1955).

Schinzel, A. Sur la décomposition des nombres naturels en sommes de nombres triangulaires distincts. Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 409–410 (1955).

A recent result of Richert [Norsk Mat. Tidsskr. **31**, 120–122 (1949); MR **11**, 646] states that every integer >33 is a sum of distinct triangular numbers. The purpose of the note under review is to show that every integer >51 is in fact the sum of four distinct triangular numbers >0. This follows from two theorems of Pall [Amer. Math. Monthly **40**, 10–18 (1933), p. 13] about the representation of numbers as sums of four distinct squares. Incidentally, a conjecture of Pall that for $n>0$, $16n+2$ is the sum of three positive squares is shown to fail for $n=8$. *D. H. Lehmer.*

Lambek, J., and Moser, L. On the distribution of Pythagorean triangles. Pacific J. Math. **5**, 73–83 (1955).

Let $P_\Delta(n)$, $P_p(n)$ and $P_a(n)$ denote respectively the number of primitive Pythagorean triangles whose hypotenuses, perimeters and areas do not exceed n. D. N. Lehmer [Amer. J. Math. **22**, 293–335 (1900)] showed that

$$P_\Delta(n)\sim n/(2\pi),\quad P_p(n)\sim n\log 2/\pi^2.$$

Recently the reviewer proved that

$$(1)\qquad P_p(n)=n\log 2/\pi^2+O(n^{1/2}\log n)$$

[Bull. Amer. Math. Soc. **54**, 1185–1190 (1948); MR **10**, 431]. W. P. Whitlock [Scripta Math. **9**, 155–161 (1943); MR **5**, 199] conjectured that $P_a(n)\sim\frac{1}{2}n^{1/2}$. The inspection of an unpublished table of F. L. Miksa showed the authors that $P_a(n)$ behaves more like $.53n^{1/2}$. By a uniform method the authors prove not only (1) but

$$(2)\qquad \begin{aligned} P_\Delta(n)&=n/(2\pi)+O(n^{1/2}\log n),\\ P_a(n)&=cn^{1/2}+O(n^{1/3}),\end{aligned}$$

where $c=\{\Gamma(1/4)\}^2/(2\pi^3)^{1/2}=.531340$. All three problems are replaced, as usual, by that of determining the number of lattice points in appropriate regions of the plane.

In the case of $P_a(n)$ the region is $xy(x^2-y^2)<n$, $0<x<y$. The number of lattice points in this region is found to be

$$L(n)=n^{1/2}\{\Gamma(1/4)\}^2/(32\pi)^{1/2}+O(n^{1/3})$$

from which (2) follows easily. It is conjectured that

$$P_a(n)=cn^{1/2}-c'n^{1/3}+O(n^{1/3}),$$

where c' is about .295. *D. H. Lehmer* (Berkeley, Calif.).

Wild, Roy E. On the number of primitive Pythagorean triangles with area less than n. Pacific J. Math. **5**, 85–91 (1955).

Referring to the preceding review, the author proves the conjecture about $P_a(n)$, namely that the number of primitive Pythagorean triangles whose area does not exceed n is

$$P_a(n)=cn^{1/2}-c'n^{1/3}+O(n^{1/4}\log n)$$

where

$$c=\{\Gamma(1/4)\}^2/(2\pi^3)^{1/2},\quad c'=-\frac{(1+2^{-1/3})\zeta(1/3)}{(1+4^{-1/3})\zeta(4/3)}=.297.$$

The author considers the number of lattice points in the first quadrant and under the curve $xy(x+y)(2x+y)=n$. This transformed region is much more like the classical hyperbolic region $xy\leqq n$ and by a judicious choice of a rectangle with one corner in the origin an estimate with an error of only $O(n^{1/4})$ for the number of lattice points is found via the Euler-MacLaurin formula. *D. H. Lehmer.*

Roth, K. F., and Szekeres, G. Some asymptotic formulae in the theory of partitions. Quart. J. Math., Oxford Ser. (2) **5**, 241–259 (1954).

The authors develop an asymptotic formula for $p_u(n)$, the number of partitions of an integer n into distinct members of an eventually increasing sequence of integers $u_1, u_2, \cdots$, subject only to the conditions:

(1) $$s=\lim_{k\to\infty}\log u_k/\log k$$

exists;

(2) $$\inf_\alpha\left\{(\log k)^{-1}\sum_{r=1}^{k}\|u_r\alpha\|^2\right\}\to\infty \quad \text{with} \quad k\to\infty,$$

where $\|\theta\|$ is the distance of θ to the nearest integer, and $\frac{1}{2}u_k<\alpha\leqq\frac{1}{2}$. (It is shown that, among others, the choices $u_k=p_k$, as well as $u_k=f(k)$, $f(p_k)$ are allowable, p_k being the kth prime, and f being a rather general polynomial.) Their formula is:

$$p_u(n)=\left[2\pi\sum_{r=1}^{\infty}u_r^2e^{\eta u_r}(1+e^{\eta u_r})^{-2}\right]^{-1/2}\times\exp\left[\sum_{r=1}^{\infty}\left\{\frac{\eta u_r}{1+e^{\eta u_r}}+\log(1+e^{-\eta u_r})\right\}\right]\times\left\{1+\sum_{r=1}^{m-2}D_r+O(n^{-(m-1)/(s+1)+\delta})\right\},$$

where $\delta>0$, m is a fixed integer >1, η is determined from $n=\sum_1^\infty u_r(1+e^{\eta u_r})^{-1}$, and the D's are defined by certain complicated summations. This formula is proved by a saddle-point method similar to that used in an earlier paper by Szekeres [same J. (2) **4**, 96–111 (1953); MR **15**, 201]. From this formula the authors deduce:

$$\log p_u(n)=\pi\left(\frac{2}{3}\frac{n}{\log n}\right)^{1/2}\left\{1+O\left(\frac{\log\log n}{\log n}\right)\right\}$$

for the number of partitions into distinct primes, as well as a more complicated formula for partitions into distinct powers. They establish also the result: there exists an n_0 and $c>0$, depending only on δ and the sequence $\{u\}$, such that $p_u(n+1)-p_u(n)>cn^{-s/(s+1)-\delta}p_u(n)$ for $n>n_0$. In the last section the restriction that the summands should be distinct is dropped; an asymptotic formula is obtained.

J. Lehner (Los Alamos, N. M.).

Ricci, Giovanni. Sulla partizione degli interi in addendi primi col procedimento del residuo minimo. Boll. Un. Mat. Ital. (3) **10**, 1–10 (1955).

By a partition of n into primes proceeding by least remainder is meant a partition $n=q_1+q_2+\cdots+q_k$ in which q_1 is down as the greatest prime $\leqq n$, q_2 as the greatest prime $\leqq n-q_1$, etc. If need be, q_k is chosen to be unity. This leads to an unique representation of n. The number k of terms is denoted by $k(n)$ and the first n to require k terms in its partition is denoted by μ_k. Thus $\mu_1=1$, $\mu_2=4$, $\mu_3=27$, $\mu_4=1354$ and μ_5 is unknown but certainly $>10^7$. The author proves two theorems. (I) Let $\epsilon>0$; then for all $k>k_0(\epsilon)$

$$(\theta+\epsilon)^{-1}\log\mu_{k-1}<\log\mu_k<(1-\epsilon)^{-1}\mu_{k-1},$$

where θ is the lower bound of all ϕ for which $p_{n+1}-p_n<p_n^\phi$, p_n being the nth prime. (II) Let $0<\delta<1$ and $\epsilon>0$; then the number $P=P(\xi,\delta)$ of integers n for which $k(n)=2$ and $(1-\delta)\xi<n\leqq\xi$, satisfies the inequalities

$$(1-\epsilon)K\delta\xi/(\log\log\xi)<P<(1+\epsilon)\delta\xi/(\log\log\xi)$$

for all sufficiently large ξ and for a certain constant $K>1/32$ depending on δ. *D. H. Lehmer* (Berkeley, Calif.).

Erdös, P. On the product of consecutive integers. III. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indagationes Math. **17**, 85–90 (1955).

It has been conjectured for a long time that the product of k consecutive integers is never an mth power ($m>1$). In this paper the author proves that there exists a constant c such that the conjecture is correct for $k>c$. The proof is entirely elementary. The author states that an earlier (unpublished) proof of this fact by Siegel and himself had made use of the well known theorem of Thue-Siegel.

H. W. Brinkmann (Swarthmore, Pa.).

Miller, J. C. P., and Woollett, M. F. C. Solutions of the Diophantine equation $x^3+y^3+z^3=k$. J. London Math. Soc. **30**, 101–110 (1955).

At the suggestion of L. J. Mordell, the authors have programmed the EDSAC to search for small sums of three integer cubes. The results are tabulated and give all primitive nontrivial solutions of

(1) $$x^3+y^3+z^3=k$$

with $|k|\leqq100$, $|z|\leqq|y|\leqq|x|\leqq3164$, in all 345 solutions. The number of solutions for a given k is highly irregular. For $k=9n\pm4$ the equation is impossible. There are 21 such solutions for $k=1$ and only 2 for $k=3$, namely (1, 1, 1) and $(-5, 4, 4)$. A further search up to $|x|=10000$ is planned for the case $k=3$. In almost all cases where the number of solutions is small, the solutions themselves are small, giving support to the conjecture that, for some k, (1) has only a finite number of solutions. No primitive solutions were found for $k=24, 30, 33, 39, 42, 52, 74, 75, 80, 84, 87, 96$. All solutions found for $k=2$ are given by

$$x=6t^3+1,\quad y=-6t^3+1,\quad z=-6t^2.$$

There are many solutions when k is itself a cube.

D. H. Lehmer (Berkeley, Calif.).

Mordell, L. J. On an infinity of integer solutions of $ax^3+ay^3+bz^3=bc^3$. J. London Math. Soc. **30**, 111–113 (1955).

The author remarks on the table of Miller and Woollett [see preceding review] that the case $k=1$ and more generally $k=c^3$ give rise to many solutions of $x^3+y^3+z^3=k$. He then proceeds to show that the more general equation in the title has an infinity of non-trivial solutions. This is done by writing the equation in the form

$$a(x+y)(x^2-xy+y^2)=b(c-z)(c^2+cz+z^2).$$

After a change and specialization of variables and a cancellation of a non-zero factor the condition becomes an indeterminate equation of the second degree of hyperbolic type having an infinity of solutions. The more general equation $xF(x, y, z)=yG(x, y, z)$, where F, G are quadratic polynomials, as discussed briefly. *D. H. Lehmer.*

Lehmer, Emma. On the number of solutions of $u^k+D\equiv w^2 \pmod p$. Pacific J. Math. **5**, 103–118 (1955).

Let $N_k(D)$ denote the number of solutions of the congruence $u^k+D\equiv w^2 \pmod p$ and let

$$\phi_k(D)=\sum_{u=1}^{p-1}\left(\frac{u}{p}\right)\left(\frac{u^k+D}{p}\right).$$

Improving a well-known result of Jacobsthal, the author shows that if $p\equiv x^2+4y^2$, $x\equiv y\equiv 1 \pmod 4$, then

$$\phi_2(D)=\begin{cases}-2x\left(\dfrac{m}{p}\right) & (D\equiv m^2 \pmod p),\\ -4y\left(\dfrac{m}{p}\right) & (D\equiv 2m^2 \pmod p);\end{cases}$$

$$N_4(D)=\begin{cases}p-1-2x\left(\dfrac{m}{p}\right) & (D\equiv m^2 \pmod p)\\ p-1-4y\left(\dfrac{m}{p}\right) & (D\equiv 2m^2 \pmod p).\end{cases}$$

Similar improvements are obtained for results of Chowla [Proc. Nat. Acad. Sci. U. S. A. **35**, 244–246 (1949); MR **10**, 592] and Schrutka [J. Reine Angew. Math. **140**, 252–265 (1911)] and related theorems. The paper closes with a table of the cyclotomic numbers $(i, j)_8$. *L. Carlitz.*

Prachar, K. Über die Summe $\sum_{p\le N}\vartheta(\alpha p)$. Monatsh. Math. **59**, 43–44 (1955).

Let a, q, N be positive integers, $1<q<N$, $(a, q)=1$, and let the real number α satisfy $|\alpha-aq^{-1}|\le q^{-2}$. Vinogradov [Trudy Mat. Inst. Steklov. **23** (1947); MR **10**, 599; **15**, 941] proved that

$$1)\quad \Big|\sum_{p\le N}\exp(2\pi i\alpha p)\Big|<CN(\log N)^{9/2}\{(q^{-1}+qN^{-1})^{\frac{1}{2}}+H^{-1}\},$$

where p runs through the primes $\le N$, and

$$H=\exp\{\tfrac{1}{3}(\log N)^{\frac{1}{2}}\}.$$

The author shows that the result can be refined if a certain part of Vinogradov's proof is replaced by an argument based on an inequality of the reviewer, concerning the number of integers composed of small primes only [Nederl. Akad. Wetensch. Proc. Ser. A. **54**, 50–60 (1951); MR **13**, 724]. The result is that in (1) H can be replaced by

$$H_1=\exp\{(\tfrac{1}{3}-\epsilon)(\log N\log\log N)^{\frac{1}{2}}\}.$$

N. G. de Bruijn (Amsterdam).

Parodi, Maurice. Fonction ζ de Riemann et nombres de Bernoulli. C. R. Acad. Sci. Paris **240**, 1395–1396 (1955).

The following formula is obtained:

$$\zeta(2, s+1)=\sum_{k=1}^{\infty}\frac{1}{k}\,\frac{(k-1)!}{(s+1)\cdots(s+k)}\qquad (s>0).$$

[cf. K. Knopp, Theorie und Anwendungen der unendlichen Reihen, 3rd ed., Springer, Berlin, 1931, p. 274].

L. Carlitz (Durham, N. C.).

Hsu, L. C. Note on an abstract inversion principle. Proc. Edinburgh Math. Soc. (2) **9**, 71–73 (1954).

A generalization of Bell's inversion principle [Duke Math. J. **15**, 79–85 (1948); MR **10**, 591] obtained by introducing the inverse of one of Bell's transfer operators T. The statement of the inversion principle remains the same, but $P(T)$ now denotes a polynomial in negative as well as positive powers of T. *J. Lehner* (Los Alamos, N. Mex.).

Fried, Ervin. On linear combinations of roots. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 155–162 (1954). (Hungarian)

The author proves the following theorem: Let K be a field; put

$$(1)\qquad c_1a_1^{1/n_1}+c_2a_2^{1/n_2}+\cdots+c_ka_k^{1/n_k}=\alpha\neq 0,$$

where the n's are integers and the a_ν and c_ν are in K. Further assume that α is not the sum of fewer than k terms of the form $c_\nu a_\nu^{1/n_\nu}$. Then if $K(a_1^{1/n_1}, \cdots, a_k^{1/n_k})$ does not contain any root of unity different from ± 1, we have $a_\nu^{1/n_\nu}\in K(\alpha)$ for $1\le\nu\le k$. [For the literature on the problem see a previous paper by Obláth, C. R. 1ier Congrès Math. Hongrois, 1950, Akad. Kiadó, Budapest, 1952; MR **15**, 11.]

Answering a question of Sierpiński the author determines the degree over the rational numbers of

$$\alpha=1+2^{1/2}+3^{1/3}+\cdots+n^{1/n},$$

where all the roots are taken as positive numbers. The degree in question equals $\prod_{p<n,\ p\text{ prime}}p^{\mu(p)}$, $\mu(p)=\sum_{k=1}^{\infty}\pi(n/p^k)+\delta_p$, where $\pi(x)$ denotes the number of primes $\le x$ and $\delta_p=0$ if $p\mid[\log n/\log p]$ and is 1 otherwise. *P. Erdős.*

Knobloch, Hans-Wilhelm. Zum Hilbertschen Irreduzibilitätssatz. Abh. Math. Sem. Univ. Hamburg **19**, 176–190 (1955).

Dörge [Math. Ann. **95**, 84–97, 247–256 (1925)] proved that if $f(x_1, \cdots, x_k, t)$ is an irreducible polynomial with integral coefficients and if $R(N)$ is the number of integers τ such that $|\tau|<N$ and $f(x_1, \cdots, x_k, \tau)$ is reducible then $R(N)\le CN^{1-\alpha}$, where $\alpha>0$. Making use of the notion of 'thin' sets, the following generalization is proved. Let $f(x_1, \cdots, x_k, t_1, \cdots, t_s)$ be an irreducible polynomial with integral coefficients in the variables $x_1, \cdots, x_k, t_1, \cdots, t_s$ and let $R(N)$ denote the number of s-tuples $(\tau_1, \cdots, \tau_s)$ of integers, $|\tau_i|\le N$, such that $f(x_1, \cdots, x_k, \tau_1, \cdots, \tau_s)$ is reducible. Then $R(N)\le CN^{s-\alpha}$, where α, C are certain positive constants. In particular, this theorem sharpens a result of van der Waerden [ibid. **109**, 13–16 (1934)] on the distribution of polynomials $a_nx^n+\cdots+a_0$, $|a_i|\le N$, with affect. Reference is also made to a paper of Specht [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 139–146 (1952); MR **14**, 251]. *L. Carlitz* (Durham, N. C.).

Krasner, Marc. Prolongement analytique dans les corps valués complets: domaines quasi connexes. C. R. Acad. Sci. Paris **238**, 2385–2387 (1954).

Krasner, Marc. Prolongement analytique dans les corps valués complets: éléments analytiques, préliminaires du théorème d'unicité. C. R. Acad. Sci. Paris **239**, 468–470 (1954).

Krasner, Marc. Prolongement analytique dans les corps valués complets: démonstration de la loi d'unicité: fonctions analytiques uniformes. C. R. Acad. Sci. Paris **239**, 745–747 (1954).

Let K be a valuated field whose value group is dense in the group of reals. The author calls a subset $D \subset K$ quasi-connected if, for each pair of points $\alpha, \xi \in D$, the distance $|x-\alpha|$ from α to a variable point x not in D takes only a finite number of values less than $|\xi-\alpha|$. Extending the notion of quasi-connectivity to subsets of the projective line $K'=K \cup \{\infty\}$, he shows it is invariant under homographies $x \to (ax+b)(cx+d)^{-1}$. Finite intersections and arbitrary unions of non-disjoint quasi-connected sets are quasi-connected.

Assuming K algebraically closed, the author defines an analytic element to be a K-valued function on a quasi-connected set D which is a uniform limit of rational functions without poles on D. He then proves the uniqueness theorem: two analytic elements which coincide on a set having a limit point within their common domain of definition coincide whenever they are both defined. *J. Tate.*

Leopoldt, H. W. Über Einheitengruppe und Klassenzahl reeller abelscher Zahlkörper. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1953**, no. 2, 48 pp. (1954).

This paper contains a continuation of H. Hasse's monograph, Über die Klassenzahl abelscher Körper [Akademie-Verlag, Berlin, 1952; MR **14**, 141], in which reductions of the class number of an abelian field K' over the rationals P were set up, thus generalizing the work of Kummer. Hasse showed [loc. cit.] that, employing concepts and methods of the class-field theory such as splitting of a character into primary characters, the main goal of the determination of the class number for K' is the detailed investigation of the class number for the maximal real subfield K of K'. The author carries out Hasse's program to a considerable extent; he separates the class number of K into components which depend on (i) the arithmetic structure of the Galois group of K/P, and (ii) intrinsic arithmetic and analytic properties of the field K and the subfields $K(\bar{\chi})$ determined by the classes $\bar{\chi}$ of algebraically conjugate characters of the characters χ for K/P. This program required the author to introduce, among others, the concept of proper $\bar{\chi}$-relative units $E^+(\bar{\chi}) \in K(\bar{\chi})$, i.e., units lying in the subfield $K(\bar{\chi})$ whose norms relative to all subfields of $K(\bar{\chi})$ are 1. Then the composite E^{K+} of all $E^+(\bar{\chi})$ is, to within ± 1, the direct product of these components. As a consequence of this fact and reductions of class numbers to suitably defined regulators, the author can show, among many other results, that the class number of K equals $\{[E_K : E^{K+}]Q^{-1}\} \prod_{\bar{\chi}} h(\bar{\chi})$ where E_K denotes the unit group of K, Q the index of the integral grouping of G augmented by the idempotent of the identical representation in the maximal order of the rational group ring of G, and integers $h(\bar{\chi})$ which depend solely on G and are norms of divisors in the fields of the simple characters belonging to $\bar{\chi}$. In this manner further complicated reductions depending on the splitting of the discriminant divisors of K'/P may be achieved.

O. F. G. Schilling (Chicago, Ill.).

Roquette, Peter. Zur Theorie der Konstantenerweiterungen algebraischer Funktionenkörper. Konstruktion der Koordinatenkörper von Divisoren und Divisorenklassen. Abh. Math. Sem. Univ. Hamburg **19**, 269–276 (1955).

Suppose that K is a function field of one variable over the perfect coefficient field Ω. Let Ω^* denote an arbitrary fixed extension of Ω and suppose that Λ/Ω is a subfield of Ω^*. Consider the natural injection mapping $\lambda(\Omega', \Omega; \mathfrak{Z}')$ of the divisors $\mathfrak{Z}'$ of $K\Omega'$ into the group of divisors $\{\mathfrak{Z}\}$ of $K\Omega^*$. Using the theory of prolongations of valuations, the author asserts the existence of a unique smallest subfield $\Omega(\mathfrak{Z})/\Omega$ of Ω^* such that $K\Omega(\mathfrak{Z})$ contains a divisor $\mathfrak{Z}'$ with $\lambda(\Omega(\mathfrak{Z}), \Omega^*; \mathfrak{Z}') = \mathfrak{Z}$. Assuming that K/Ω contains a prime divisor of degree 1, an application of theorem of Riemann-Roch is shown to yield a similar result of divisor classes of $K\Omega^*$. [See also W. L. Chow, Proc. Amer. Math. Soc. **1**, 797–799 (1950), where the generalization of the author's result is established for arbitrary algebraic varieties using the methods of the Chow-van der Waerden associated forms; MR **12**, 740.] *O. F. G. Schilling* (Chicago, Ill.).

Kawada, Y., and Tate, J. On the Galois cohomology of unramified extensions of function fields in one variable. Amer. J. Math. **77**, 197–217 (1955).

Let k be a field of algebraic functions in one variable, with an algebraically closed field of constants k_0. Let K/k be a finite Galois extension with Galois group G, and assume that K/k is everywhere unramified. Let U be the group of all elements of finite order of the multiplicative group of k_0. Let $E=E(K)$ denote the group of the divisor classes of K, and define its dual E' as the group of all homomorphisms of E into U. The Galois group G operates on E and E' in a natural fashion.

By investigating the cohomology groups $H^r(G, E')$, where r is an arbitrary integer, positive, negative, or 0, the authors establish a theory of extensions K/k as described above which is exactly analogous to class field theory, and in which the group E' takes the place of the idèle class group. In particular, they prove that there is a canonical homomorphism of $H^{r-2}(G, Z)$ (where Z is the additive group of the integers) onto $H^r(G, E')$, which is an isomorphism whenever the characteristic of k does not divide the order of G. Moreover, just as in class field theory, this homomorphism results from cup multiplication with a canonical generator of $H^2(G, E')$, the fundamental class, with the same restriction and inflation properties in field towers which hold in class field theory.

The usual divisor class norm map $E(K) \to E(k)$ yields the dual conorm N': $E'(k) \to E'(K)$ which is shown to be an isomorphism (into), and which plays the part analogous to the usual injection map of idèle classes. In particular, $N'(E'(k))$ coincides with the G-fixed part $E'(K)^G$ of $E'(K)$. On the other hand, the ordinary injection of $E(k)$ into $E(K)$ dualizes to a homomorphism N: $E'(K) \to E'(k)$ which takes the place of the idèle class norm; indeed, for any x in $E'(K)$, $N'N(x)$ is the product of the G-conjugates of x in $E'(K)$. It follows that $H^0(G, E')$ is canonically isomorphic with $E'(k)/N(E'(K))$. The above general cohomology result, specialized to $r=2$, gives a canonical homomorphism of G onto $E'(k)/N(E'(K))$ which is an isomorphism whenever G is abelian and of an order prime to the characteristic of k. This isomorphism (more generally, also the homomorphism) is made explicit by using a standard 2-cocycle representing the fundamental class. The result provides a link between this pseudo class field theory and the Kummer theory of

unramified abelian extensions of degree prime to the characteristic. In particular, the Kummer theory now yields the appropriate existence theorem for the pseudo class field theory.

The second part of the paper is concerned with the classical case, where k_0 is the field of the complex numbers. In that case, the group E_0 of the divisor classes of degree 0 is isomorphic with the $2g$-dimensional torus group T^{2g}, where g is the genus of K, and T is the multiplicative group of all unimodular complex numbers. Accordingly, E_0 may be equipped with the structure of a compact topological group. Also, E/E_0 is isomorphic with Z, and E is given the structure of a locally compact topological group such that E/E_0 is discrete and the induced topology on E_0 coincides with that obtained from the isomorphism with T^{2g}. It is shown first that the group E' of the above abstract theory may here be replaced by the group E^s of all continuous homomorphisms of E into T without losing any of the former results. Then the various elements of the abstract theory are interpreted in topological and analytical terms involving the Riemann surface $R(K)$ of K.

Thus the elements of E^s are represented by functions on the divisors of K whose values are obtained by integrating certain differentials (attached uniquely to the divisors of degree 0) over 1-cycles of $R(K)$. By analyzing the connections between the differentials and homology classes of K and those of k, it is then shown that $E'(k)/N(E'(K))$ is isomorphic with $H_1(k)/p(H_1(K))$, the factor group of the 1-dimensional homology group $H_1(k)$ of $R(k)$ modulo the image under the projection map p of the 1-dimensional homology group $H_1(K)$ of $R(K)$. On the other hand, if F stands for fundamental group, the Galois group G of K/k is isomorphic with $F(k)/p(F(K))$. If G is abelian this gives the result that G is isomorphic with $H_1(k)/p(H_1(K))$. Using an explicit 2-cocycle representing the fundamental class of $H^2(G, E')$ it is finally shown that the reciprocity isomorphism of the pseudo class field theory coincides with the composite of the two isomorphisms:

$$G \approx H_1(k)/p(H_1(K)) \approx E'(k)/N(E'(K)).$$

G. Hochschild (Urbana, Ill.).

Siegel, Carl Ludwig. Indefinite quadratische Formen und Funktionentheorie. I. Math. Ann. 124, 17–54 (1951).

Let $\mathfrak{S}[\mathfrak{x}]=\mathfrak{x}'\mathfrak{S}\mathfrak{x}=\sum_{k,l=1}^{n} s_{kl}x_kx_l$ be a quadratic form in m variables $x_1, x_2, \cdots, x_m$, where the matrix $\mathfrak{S}$ is supposed to be real, symmetric ($\mathfrak{S}'=\mathfrak{S}$) and non-singular. If its signature is $(n, m-n)$, there exists a real linear transformation $\mathfrak{x}=\mathfrak{C}\mathfrak{y}$, transforming $\mathfrak{S}[\mathfrak{x}]$ into $q-r$, where $q=y_1^2+\cdots+y_n^2$, $r=y_{n+1}^2+\cdots+y_m^2$. The positive definite quadratic form $\mathfrak{P}[\mathfrak{x}]$ defined by $\mathfrak{P}[\mathfrak{x}]=q+r$ (and whose matrix is $(\mathfrak{C}\mathfrak{C}')^{-1}$) is called a "majorant" of $\mathfrak{S}[\mathfrak{x}]$. All majorants of $\mathfrak{S}$ are exactly the solutions of $\mathfrak{P}\mathfrak{S}^{-1}\mathfrak{P}=\mathfrak{S}$, $\mathfrak{P}'=\mathfrak{P}>0$. A parametric representation of the set P of all majorants $\mathfrak{P}$ of S shows that P is in birational correspondence with those $(mn-n-1)$-dimensional linear subspaces of $(m-1)$-dimensional projective space (in which the coordinates $x_1, x_2, \cdots, x_m$ are the components of the column vector x) in all points of which $\mathfrak{S}[\mathfrak{x}]$ is everywhere positive, so that P corresponds to the subset determined by $\mathfrak{S}[\mathfrak{x}]>0$ of the Grassmann variety of $n(m-n)$ dimensions. If Ω is the "orthogonal" group of $\mathfrak{S}$ (the group of all linear transformations which transform $\mathfrak{S}[\mathfrak{x}]$ into itself) then a transitive representation of Ω on P is obtained by considering the mappings $\mathfrak{P}\to\mathfrak{P}[\mathfrak{V}]=\mathfrak{V}'\mathfrak{P}\mathfrak{V}$ (where $\mathfrak{V}\in\Omega$), which map P into itself. A Riemannian metric on P, invariant under Ω, is then defined. If $\mathfrak{S}$ is supposed to be rational and Γ is the subgroup of Ω consisting of all "units" of $\mathfrak{S}$ (i.e. integral matrices $\mathfrak{V}$ satisfying $\mathfrak{S}[\mathfrak{V}]=\mathfrak{S}$) then Γ is discontinuous on P and the author shows by means of the reduction theory of quadratic forms that the volume of the fundamental domain F_0 of Γ on P is, with one trivial exception), always finite.

A quadratic equation with integral coefficients can be written in the form $\mathfrak{S}[\mathfrak{x}+\mathfrak{a}]=t$, where $\mathfrak{S}=(s_{kl})$ is a given symmetric matrix, $\mathfrak{a}$ a given vector and t a given number, and where s_{kk}, $2s_{kl}$, $2\mathfrak{S}\mathfrak{a}$, $\mathfrak{S}[\mathfrak{a}]-t$ are integral. If $\mathfrak{P}$ is a majorant of $\mathfrak{S}$, $z=x+iy$ a complex variable with $y>0$ and if $\mathfrak{R}=x\mathfrak{S}+iy\mathfrak{P}$, the author introduces the theta series

$$f(z)=f_{\mathfrak{a}}(z)=f_{\mathfrak{a}}(z, \mathfrak{P})=\sum_{\mathfrak{x}} e^{\pi i(\mathfrak{R}[\mathfrak{x}+\mathfrak{a}])},$$

where the summation extends over all integral vectors $\mathfrak{x}$. Since the imaginary part of $\mathfrak{R}$ is $y\mathfrak{P}$ and $y>0$ the series is convergent. The function $f(z)$ is not an analytic function of z. Under modular substitutions it shows a behaviour which is similar to that of the theta series for positive definite quadratic forms. All those automorphs $\mathfrak{U}$ of $\mathfrak{S}$ such that $\mathfrak{U}\mathfrak{a}-\mathfrak{a}$ is integral constitute a subgroup $\Gamma_{\mathfrak{a}}$ of Γ. The function $f_{\mathfrak{a}}(z, \mathfrak{P})$ remains unchanged if $\mathfrak{P}$ is replaced by $\mathfrak{P}[\mathfrak{U}]$. The author's principal result is the formula

$$\text{(I)}\qquad V^{-1}\int_F f_{\mathfrak{a}}(z, \mathfrak{P})dv=\gamma+\sum_r \gamma(r)(z-r)^{-\frac{1}{2}n}(\bar{z}-r)^{\frac{1}{2}(n-m)}.$$

Here the integration is extended over the fundamental domain F of $\Gamma_{\mathfrak{a}}$ and V is the volume of F which is shown to be finite by means of the Minkowski reduction theory of quadratic forms; $\gamma=1$ or 0 according as $\mathfrak{a}$ is integral or not; r runs through all rational numbers a/b (where $(a, b)=1$) and

$$\gamma(r)=e^{\frac{1}{4}\pi i(2n-m)}s^{-\frac{1}{2}}b^{-n}\sum_{\mathfrak{y} \bmod b} e^{2\pi i r\mathfrak{S}[\mathfrak{y}]};$$

s is the determinant of $2\mathfrak{S}$; $\mathfrak{y}$ runs through a complete set of residues mod b. It is supposed that $n(m-n)>0$, $n>3$ and for $m=4$ the zero forms $\mathfrak{S}[\mathfrak{x}]$ for which the determinant $|\mathfrak{S}|$ is a square, are excluded.

Since $\mathfrak{S}[\mathfrak{x}]$ is an indefinite form, the number of units is infinite. Therefore if one solution $\mathfrak{y}_1=\mathfrak{x}_1+\mathfrak{a}$ of the diophantine equation $\mathfrak{S}[\mathfrak{x}+\mathfrak{a}]=t$ exists, there is an infinity of associated solutions given by $\mathfrak{y}=\mathfrak{U}\mathfrak{y}_1$, where $\mathfrak{U}\in\Gamma_{\mathfrak{a}}$. However, the number of classes of associated solutions is finite. Every solution $\mathfrak{y}$ is provided with a mass $\mu(\mathfrak{S}, \mathfrak{y})$ depending on its class only. The "Lösungsmass" ("number" of solutions) is defined as the sum

$$M(\mathfrak{S}, \mathfrak{a}, t)=\sum\mu(\mathfrak{S}, \mathfrak{y})$$

extended over a complete set of solutions $\mathfrak{y}=\mathfrak{x}+\mathfrak{a}\neq 0$ of $\mathfrak{S}[\mathfrak{y}]=t$ which are non-associated with respect to $\Gamma_{\mathfrak{a}}$. From the Fourier-series development of the Eisenstein series on the right-hand side of (I) the author obtains his second fundamental result:

$$M(\mathfrak{S}, \mathfrak{a}, t)=\mu_{\mathfrak{a}}(\mathfrak{S})\prod_p \delta_p(\mathfrak{S}, \mathfrak{a}, t),$$

where p runs through all primes; $\mu_{\mathfrak{a}}(\mathfrak{S})$ is the volume of the fundamental domain of $\Gamma_{\mathfrak{a}}$ on Ω, which is connected with the volume V on P by means of the formula

$$\mu_{\mathfrak{a}}(S)=\rho_n\rho_{m-n}S^{-\frac{1}{2}(m+1)}V;$$

S is the absolute value of the determinant of $\mathfrak{S}$ and

$$\rho_l=\prod_{k=1}^{l}\pi^{-\frac{1}{2}k}\Gamma^{-1}(\tfrac{1}{2}k).$$

The numbers $\delta_p(\mathfrak{S}, \mathfrak{a}, t)$ are the p-adic solution densities defined by

$$\delta_p(\mathfrak{S}, \mathfrak{a}, t) = \lim_{k \to \infty} \frac{A_q(\mathfrak{S}, \mathfrak{a}, t)}{q^{m-1}} \quad (q = p^k),$$

where $A_q(\mathfrak{S}, \mathfrak{a}, t)$ is for any integer q the number of solutions $\mathfrak{x}$ of the congruence

$$\mathfrak{S}[\mathfrak{x}+\mathfrak{a}] \equiv t \pmod{q}.$$

H. D. Kloosterman (Leiden).

Siegel, Carl Ludwig. Indefinite quadratische Formen und Funktionentheorie. II. Math. Ann. **124**, 364–387 (1952).

The author shows that his results on indefinite quadratic forms [see the preceding review] with rational integral coefficients can be extended to the case of algebraic number fields and even to certain algebras of finite rank over the field of rational numbers. He develops in detail the case of the Gaussian field (of all numbers $a+ib$, where a and b are rational). In order to extend the notion of quadratic form to the case of an algebra A the existence of an involution $a \to a^*$ such that $a^{**}=a$, $(a+b)^*=a^*+b^*$, $(ab)^*=b^*a^*$ is supposed, in which case a quadratic form is defined by $\mathfrak{x}^*\mathfrak{S}\mathfrak{x} = \sum_{k,l=1}^{n} x_k^* s_{kl} x_l$, $s_{lk} = s^*_{kl}$. The algebra A is further supposed to be simple. *H. D. Kloosterman* (Leiden).

Koecher, Max. Über Thetareihen indefiniter quadratischer Formen. Math. Nachr. **9**, 51–85 (1953).

Let $\mathfrak{S} = (s_{kl})$ be a real symmetric matrix with m rows and columns and let the s_{kk}, $2s_{kl}$ be integers. The corresponding quadratic form is supposed to be indefinite and of signature $(\mu, m-\mu)$, so that $\mu(m-\mu) > 0$. Let $\mathfrak{P}$ be a "majorant" of $\mathfrak{S}$, satisfying $\mathfrak{P}\mathfrak{S}^{-1}\mathfrak{P} = \mathfrak{S}$, $\mathfrak{P}' = \mathfrak{P} > 0$. If further $\mathfrak{A}$ is a rational matrix with m rows and n columns ($n \leqq m$) and such that $2\mathfrak{S}\mathfrak{A}$ is an integral matrix, and if $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are variable complex symmetric matrices with n rows and columns, the author considers theta series of the following type:

$$f_{\mathfrak{A}}(\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{P}) = \sum_{\mathfrak{B} \equiv \mathfrak{A}(1)} \exp \pi i \sigma\{\mathfrak{S}[\mathfrak{B}](\mathfrak{Z}_1+\mathfrak{Z}_2) + \mathfrak{P}[\mathfrak{B}](\mathfrak{Z}_1-\mathfrak{Z}_2)\},$$

where the summation is extended over all rational matrices $\mathfrak{B}$ which are congruent mod 1 with the given matrix $\mathfrak{A}$, and where $\sigma\{\mathfrak{U}\}$ denotes the trace of the matrix $\mathfrak{U}$. The series is convergent for $\mathfrak{J}(\mathfrak{Z}_1) > 0$, $\mathfrak{J}(-\mathfrak{Z}_2) > 0$. The author studies the behaviour of the function if $\mathfrak{Z}_\nu$ ($\nu = 1, 2$) are replaced by $(\mathfrak{F}\mathfrak{Z}_\nu + \mathfrak{G})(\mathfrak{C}\mathfrak{Z}_\nu + \mathfrak{D})^{-1}$, where $\mathfrak{M} = \begin{pmatrix} \mathfrak{F} & \mathfrak{G} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix}$ is an element of Siegel's modular group M_n of degree n.

In order to eliminate the special choice of the fixed majorant $\mathfrak{P}$ the functions f are integrated over the fundamental domain on the space P of the majorants $\mathfrak{P}$ of a certain subgroup of the group of automorphs of $\mathfrak{S}$ (the group of all integral matrices which transform $\mathfrak{S}$ into itself). The resulting functions $g(\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{S}) = \int f(\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{P}) dv$ show the same behaviour under M_n as the functions f. They are solutions of certain partial differential equations of order $2n$ which in the case $n=1$ reduce to the confluent hypergeometric differential equation. *H. D. Kloosterman.*

Koecher, Max. Zur Theorie der Modulformen n-ten Grades. II. Math. Z. **61**, 455–466 (1955).

The definitions of part I [Math. Z. **59**, 399–416 (1954); MR **15**, 603] are used. The definition of a cusp-form is altered. A scalar product is obtained, thereby furnishing the set of cusp-forms with a metric. This is done for a class, congruence subgroups, of subgroups of the symplectic group, and is a generalization of results of Petersson and Maass. *H. S. Zuckerman* (Seattle, Wash.).

Braun, Hel. Der Basissatz für hermitische Modulformen. Abh. Math. Sem. Univ. Hamburg **19**, 134–148 (1955).

In some earlier papers [Ann. of Math. (2) **50**, 827–855 (1949); **51**, 92–104 (1950); **53**, 143–160 (1951); MR **11**, 333; **12**, 482] the author developed a theory of hermitian modular forms. She did not succeed at that time in proving the basis theorem stating that the rank of the linear set of hermitian modular forms of a given dimension is finite. Now using some idea of Koecher [Math. Z. **59**, 399–416 (1954); MR **15**, 603] she gives a proof of the basis theorem.

H. D. Kloosterman (Leiden).

Cohn, Harvey. Approach to Markoff's minimal forms through modular functions. Ann. of Math. (2) **61**, 1–12 (1955).

The author considers a subgroup G of genus 1 of the modular group and generated by two modular matrices A and B with a commutator $B^{-1}A^{-1}BA$ of the form $K = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$. Then if $C = (AB)^{-1}$ an identity of Fricke yields $S(A)^2+S(B)^2+S(C)^2 = S(A)S(B)S(C)$, where S denotes the trace of the matrices. The integers $S(A)$, $S(B)$, $S(C)$ must be divisible by 3. Writing $S(A) = 3a$, $S(B) = 3b$, $S(C) = 3c$, Markoff's equation $a^2+b^2+c^2 = 3abc$ is obtained. All integral solutions of this equation can be obtained from the smallest solution $a=b=c=1$ by means of the two operations which replace the triples A, B, C by BA, A^{-1}, B^{-1} and A, $A^{-1}B$, B^{-1}. The number k is shown to be ± 6. Markoff's forms (whose minima are greater in absolute value than one-third the square root of the discriminant) are obtained by associating to $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the form $a_{21}x^2 + (a_{22}-a_{11})xy - a_{12}y^2$, whose minimum and discriminant are $a = a_{12}$ and $S(A)^2 - 4 = 9a^2_{21} - 4$ respectively. The author finds new forms by taking $k = -12$. The unimodular matrices A of G are not integral in this case but $2A$ is integral. The doubled forms have a minimum which exceeds one-sixth of the square root of the discriminant.

H. D. Kloosterman (Leiden).

Newman, Morris. Structure theorems for modular subgroups. Duke Math. J. **22**, 25–32 (1955).

Let $G_0(n)$ (for any positive integer n) be the group of all modular matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ (where α, β, γ, δ are integers, $\alpha\delta - \beta\gamma = 1$) such that $\gamma \equiv 0 \pmod{n}$. The author shows that any subgroup of $G_0(n)$ which contains $G_0(mn)$ is a group $G_0(dn)$, where d divides m. He also proves: Let G_1 and G_2 be groups of modular matrices, $G_1 \subset G_2$ of finite index, and let F_1 and F_2 be the corresponding fields of modular functions. Suppose that G_2 is of genus zero so that F_2 is a simple transcendental extension $C(a)$ of the field of complex numbers. Then if b is an element of F_1 which is not invariant under any larger subgroup than G_1, we have $F_1 = F_2(b) = C(a, b)$. From these theorems he derives a number of special results of which the following is typical: If $J(\tau)$ is the modular invariant, the field of modular functions belonging to $G_0(n)$ is generated by $J(\tau)$ and $J(n\tau)$ over C.

H. D. Kloosterman (Leiden).

Mahler, Kurt. On a problem in the geometry of numbers. Rend. Mat. e Appl. (5) **14**, 38–41 (1954).

Let K be a bounded convex body, symmetric with respect to the origin, of points $(x_1, x_2, \cdots, x_n)$ in Euclidean n-space. For $s>0$, let K_s denote the convex body of all the points of K for which $|x_n| \leqq s$ and $\Delta(s)$ the critical determinant of K_s. The author conjectures that $\Delta(s)/s$ is a decreasing function of s and proves this for $n=2$. *D. Derry.*

Barnes, E. S. The inhomogeneous minimum of a ternary quadratic form. Acta Math. **92**, 13–33 (1954).

Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D \neq 0$. It was proved by the reviewer [Acta Math. **80**, 65–95 (1948); MR **10**, 101] that there exist, for any real x_0, y_0, z_0, integers x, y, z such that

$$(*) \qquad |Q(x+x_0, y+y_0, z+z_0)| \leqq \left(\frac{27}{100}|D|\right)^{1/3}.$$

It was further proved that the constant $27/100$ can be diminished slightly if Q is not equivalent to a multiple of Q_1, where

$$Q_1(x, y, z) = x^2+5y^2-z^2+5yz+zx.$$

In the present paper the author succeeds in proving more, namely that (*) can be improved to

$$|Q(x+x_0, y+y_0, z+z_0)| < \left(\frac{4}{15}|D|\right)^{1/3}$$

except in two special cases. These are the cases when $2x_0$, $2y_0$, $2z_0$ are odd integers and when in addition Q is equivalent to a multiple of either Q_1 or Q_2, where

$$Q_2(x, y, z) = 2x^2-y^2+15z^2.$$

The proof makes use of the powerful technique developed by Barnes and Swinnerton-Dyer in the paper reviewed below for investigating the best asymmetric inequalities satisfied by non-homogeneous indefinite binary quadratic forms. The details of the present paper are complicated and delicate, as might be expected. *H. Davenport* (London).

Barnes, E. S., and Swinnerton-Dyer, H. P. F. The inhomogeneous minima of binary quadratic forms. III. Acta Math. **92**, 199–234 (1954).

Delauney (Delone) showed that every inhomogeneous two-dimensional lattice ('grid') $\mathfrak{L}$ of determinant Δ with no points on the co-ordinate axes always has 4 points A, B, C, D, one in each quadrant, forming a parallelogram of determinant Δ [Izv. Akad. Nauk. SSSR. Ser. Mat. **11**, 505–538 (1947); MR **9**, 334]. The authors show that such parallelograms ('divided cells') may be arranged in a sequence with a simple algorithm relating successive cells. This they use to investigate the inhomogeneous critical determinant D_m of the region $-1<\xi\eta<m$ $(m \geqq 1)$. There are always critical lattices of the form

$$\xi = \alpha(x-\tfrac{1}{2})+\beta(y-\tfrac{1}{2}), \quad \eta = \gamma(x-\tfrac{1}{2})+\delta(y-\tfrac{1}{2})$$

(x, y run through the integers). As an application, D_m is evaluated in a neighbourhood of $m=2$ and is shown to have a very complicated structure as a function of m. An estimate of the second inhomogeneous minimum for $m=2$ is given which is better than Davenport's [Acta Math. **80**, 65–95 (1948); MR **10**, 101]. *J. W. S. Cassels.*

Barnes, E. S. The inhomogeneous minima of binary quadratic forms. IV. Acta Math. **92**, 235–264 (1954).

The author continues the discussion of the 'algorithm of the divided cell' [see the preceding review]. Every such algorithm defines a sequence of pairs of integers h_n, k_n satisfying simple restrictions and conversely every such sequence of integer pairs defines an inhomogeneous lattice $\mathfrak{L}$. The sums h_n+k_n characterise the corresponding homogeneous lattice. It is claimed that this algorithm is more effective than that used in I, II of the series [Acta Math. **87**, 259–323; **88**, 279–316 (1952); MR **14**, 730, 956]. As an illustration $M(x^2-19y^2) = 170/171$ (wrongly given in I) and $M(x^2-46y^2) = 76877/48668$ are evaluated. See also the paper reviewed below. *J. W. S. Cassels.*

Godwin, H. J. On the inhomogeneous minima of certain norm-forms. J. London Math. Soc. **30**, 114–119 (1955).

Using the notation of Barnes and Swinnerton-Dyer [Acta Math. **87**, 259–323 (1952); MR **14**, 730], the author fills in the gaps in their tables by finding the value of the inhomogeneous minimum $M_1(f_m)$ and an upper bound for $M_2(f_m)$ for $m=46$, 57, 67, 71, 86 and 94. He also evaluates $M_1(f_{73})$, $M_2(f_{73})$ and finds an upper bound for $M_3(f_{73})$. His method is a modification of the method of Barnes and Swinnerton-Dyer to make it easier to apply to norm-forms of fields with a large fundamental unit. *H. S. A. Potter.*

Hlawka, Edmund. Inhomogene Minima von Sternkörpern. Monatsh. Math. **58**, 292–305 (1954).

This paper generalizes a group of theorems found by the reviewer and afterwards proved more simply by Cassels [Proc. Cambridge Philos. Soc. **48**, 72–86, 519–520 (1952); MR **13**, 919]. Let S be a star body in n-dimensional space, symmetrical about the origin, with distance-function $f(\mathbf{x})$. The inhomogeneous minimum of S relative to a lattice Λ is defined by

$$E(S, \Lambda) = \sup_{\mathbf{x}} \inf_{\mathbf{l}} f(\mathbf{l}+\mathbf{x}),$$

where $\mathbf{l}$ is any point of Λ and $\mathbf{x}$ any point of space. The object of the paper is to obtain a lower bound for $E(S, \Lambda)$ when S is the direct product of a bounded star body S_1 in n_1 dimensions and a bounded star body S_2 in n_2 dimensions $(n_1+n_2=n)$, so that $f=(f_1^{n_1}f_2^{n_2})^{1/n}$. Let λ_1, λ_2 be positive parameters satisfying $\lambda_1^{n_1}\lambda_2^{n_2}=1$, and let S_{λ_1,λ_2} denote the (bounded) n-dimensional star body defined by

$$\max(\lambda_1 f_1, \lambda_2 f_2) \leqq 1.$$

The first general result given is that if S_1 and S_2 are convex, then

$$(*) \qquad E(S, \Lambda) \geqq c_n^{-1} \inf_{\lambda_1,\lambda_2} \mu_n(S_{\lambda_1,\lambda_2}, \Lambda),$$

where μ_n denotes the nth successive minimum of the star body S_{λ_1,λ_2} relative to Λ, and $c_n = 128n^{7/2}(n!)^5$. The earlier results of the reviewer arise as the special cases when S_1 and S_2 are either intervals in one dimension or circles in two dimensions, and n is accordingly 2 or 3 or 4. The author does not prove (*) explicitly, but proves a theorem which is still more general, in that it is postulated only that one of S_1, S_2 is convex. He introduces the polar bodies $H_1 \leqq 1$ and $H_2 \leqq 1$ of S_1 and S_2 in their respective spaces, in the sense of Mahler [Časopis Pěst Mat. Fys. **68**, 93–102 (1939); MR **1**, 202], and denotes by U_{λ_1,λ_2} the (bounded) star body defined by $\max(\lambda_1 H_1, \lambda_2 H_2) \leqq 1$. Then the more general theorem asserts an inequality similar to (*) but with $\mu_n(S_{\lambda_1,\lambda_2}, \Lambda)$ replaced by $1/\mu_1(U_{\lambda_1,\lambda_2}, \Lambda')$, where Λ' is the n-dimensional lattice reciprocal to Λ. The proof is stated to be based on the methods of Cassels. *H. Davenport.*

Davenport, H. On a theorem of Furtwängler. J. London Math. Soc. **30**, 186–195 (1955).

Let $F(x_1, \cdots, x_{n-1})$ be the distance function of a bounded star body in $n-1$ dimensions, so that F is continuous, homogeneous of degree 1, and positive except at $x_1=\cdots=x_{n-1}=0$. Let $C(F)$ be the inf of the constants C such that for any real $\theta_1, \cdots, \theta_{n-1}$ there are infinitely many approximations $q_1/q, \cdots, q_{n-1}/q$ satisfying

$$F(q\theta_1-q_1, \cdots, q\theta_{n-1}-q_{n-1})<(C/q)^{1/(n-1)}.$$

Finally, let K be the n-dimensional region defined by

$$\{F(x_1, \cdots, x_{n-1})\}^{n-1}|x_n|\leqq 1.$$

The main theorem is that $C(F)=1/\Delta(K)$, where $\Delta(K)$ is the critical determinant of K. The method is similar to, but simpler than, that devised by P. Furtwängler [Math. Ann. **96**, 169–175 (1926); **99**, 71–83 (1928)], who proved a consequence of the above theorem, namely that if

$$F(x_1, \cdots, x_{n-1})=\max(|x_1|, \cdots, |x_{n-1}|),$$

then $C(F)\geqq|d|^{-1/2}$, where d is the numerically smallest discriminant of any real algebraic number field of degree n. It is also shown that the inf of the numbers C' such that the inequality

$$|\theta_1p_1+\cdots+\theta_{n-1}p_{n-1}+p|<C'\{F(p_1, \cdots, p_{n-1})\}^{-n+1}$$

has infinitely many solutions, is again $1/\Delta(K)$.

W. J. LeVeque (Ann Arbor, Mich.).

Descombes, R. Etude diophantienne de certaines formes linéaires non homogènes. Bull. Soc. Math. France **82**, 197–299 (1954).

The author presents amplifications and detailed proofs of results announced earlier [Descombes and Poitou, C. R. Acad. Sci. Paris **234**, 581–583, 1522–1524 (1952); MR **13**, 825, 921; Descombes, ibid. **236**, 1401–1403, 1460–1462 (1953); MR **14**, 851]. The proofs use ingenious modifications of the continued fraction process but require much detailed numerical argumentation. *J. W. S. Cassels.*

ANALYSIS

Singer, Ivan. Théorèmes de moyenne pour les systèmes de fonctions. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Şti. Mat. Fiz. **5**, 251–271 (1953). (Romanian. Russian and French summaries)

For a system of real continuous functions $\varphi_0(x), \cdots, \varphi_n(x)$ on a closed and bounded interval $[a, b]$, the author considers "polynomials" of order k, that is, expressions of the form $a_0\varphi_0(x)+\cdots+a_k\varphi_k(x)$, $k\leqq n$, where the coefficients a_i are real and not all equal to zero. Notions of Lagrange interpolation polynomial for a function $f(x)$ relative to $x_1<\cdots<x_{n+1}$ in $[a, b]$, Chebyshev system, and divided difference are generalized. Results are obtained which synthesize and generalize various mean-value theorems of analysis. *E. F. Beckenbach* (Los Angeles, Calif.).

Biernacki, Mieczysław. Sur un problème de M. Leja relatif à une fonction des distances entre des points. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 113–120 (1954). (Polish and Russian summaries)

The author attributes to F. Leja the result that if $P_1, \cdots, P_n$, Q are distinct points in the Euclidean plane, then

$$\frac{QP_2\cdot QP_3\cdots QP_n}{P_1P_2\cdot P_1P_3\cdots P_1P_n}+\frac{QP_1\cdot QP_3\cdots QP_n}{P_2P_1\cdot P_2P_3\cdots P_2P_n}+\frac{QP_1\cdot QP_2\cdots QP_{n-1}}{P_nP_1\cdot P_nP_2\cdots P_nP_{n-1}}\geqq 1.$$

He also attributes the proof to Leja; this is an immediate application of the Lagrange interpolation formula for the complex values $z_1, \cdots, z_n$, and for the function $f(z)\equiv 1$. Answering a question likewise attributed to Leja, as to whether the results still holds in Euclidean spaces of higher dimensions, the author now points out that the inequality is violated in three-dimensional Euclidean space, for $n=4$, if Q is the center of gravity of a regular tetrahedron of which P_1, P_2, P_3, P_4 are the vertices; shows that, for $\epsilon>0$ and n an arbitrary integer $\geqq 2$, there exist, in three-dimensional Euclidean space, points for which the left member is $<(2e^{-1/2}-\epsilon)^{-n}$ (which $\to 0$ as $n\to\infty$); but shows that, in Euclidean space of $k\geqq 3$ dimensions, the left member is $\geqq 1/2^{n-2}$. *E. F. Beckenbach* (Los Angeles, Calif.).

*Gončarov, V. L. **Teoriya interpolirovaniya i priblizheniya funkcii. [Theory of interpolation and approximation of functions.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 327 pp. 15.95 rubles.

This book is an exposition of the classical theory of interpolation and approximation. Some of the developments of the 20 years since the first edition are briefly noted, but modern methods and the newer problems are untouched except for a brief appendix on normed linear spaces. The book is an excellent introduction to the theory, especially for a reader who is unfamiliar with the latest terminology. The discussion is quite detailed and numerous examples are worked out. Contents: Interpolation at a set of points, convergence and divergence of infinite interpolation processes; Weierstrass's theorem on uniform approximation of continuous functions; mean-square approximation and orthogonal functions; best approximation (Chebyshev, S. Bernstein, D. Jackson); interpolation and approximation in the complex domain. *R. P. Boas, Jr.*

Gončar, A. A. On best approximations by rational functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 205–208 (1955). (Russian)

The author gives several results on the best approximation $R_n(f)$ to the function f on $[0, 1]$ by rational functions of degree n. If $\phi(n)$ and $\omega(\delta)$ are positive, $\phi(n)\to 0$ as $n\to\infty$, $\omega(\delta)\to 0$ as $\delta\to 0$, there is a continuous f such that $R_n(f)<\phi(n)$, $\limsup \omega_f(\delta)/\omega(\delta)>1$, where ω_f is the modulus of continuity of f. If $R_n(f)<n^{-1-\epsilon}$, $\epsilon>0$, then $f(x)$ is almost everywhere differentiable. If f is continuous and $\{R_n(f)\}^{1/n}\to 0$, then $f(x)\equiv 0$ on some interval implies $f(x)\equiv 0$ on $[0, 1]$. More generally, it is enough to have $R_n(f)<Cq^n$, $q<1/36$, and "interval" can be replaced by "set of positive measure." *R. P. Boas, Jr.* (Evanston, Ill.).

Georgĕv, Georg. Mechanical quadratures with a minimal number of terms. Rozprawy Mat. **8**, 72 pp. (1955). (Russian. English summary)

Given a domain R in m-space, and an integer n, the problem is to find the smallest number N such that a

formula of the form

$$\int\cdots\int_R P(x_1, \cdots, x_m)dx_1\cdots dx_m = \sum_{i=1}^{N}\lambda_i P(x_1^{(i)}, \cdots, x_m^{(i)})$$

is valid for all polynomials of degree not exceeding n; in particular, the λ_i may be required to be equal. The author solves these problems, and determines all possible formulas, when m is 2 or 3 and n is 1 or 2, and also, if R has a center of symmetry, when m is 2 or 3 and n is 3. Some of the results were announced in Dokl. Akad. Nauk SSSR (N.S.) **83**, 521–524 (1952); **89**, 389–392 (1953); MR **13**, 827; **14**, 852.

R. P. Boas, Jr. (Evanston, Ill.).

Lalaguë, Pierre. Sur des classes de fonctions indéfiniment dérivables. C. R. Acad. Sci. Paris **240**, 1041–1043 (1955).

The author states several theorems on regularization and equivalence of classes of functions defined by $|f^{(n)}(x)| < c^n M_n e^{kx}$ ($x \geq 0$; k positive or negative) or by $|f^{(n)}(x)| < c^n M_n \exp(\pm k^2 x^2)$ ($-\infty < x < \infty$). He also states connections between ordinary classes $C\{M_n\}$ and the classes of the new kind obtained by the change of variable $x = \cos\theta$.

R. P. Boas, Jr. (Evanston, Ill.).

Puig Adam, P. Reduced ascendents and reduced descendents in the algorithm of continued fractions of differential elements. Rev. Acad. Ci. Madrid **48**, 11–22 (1954). (Spanish)

The author's continuous continued fraction is

$$\frac{A(x,t)z+B(x,t)}{C(x,t)z+D(x,t)} = \lim\left\{a_0+\frac{1}{b_0+}\frac{1}{a_1+}\frac{1}{b_1+}\cdots\right.$$
$$\left.\cdots\frac{1}{+a_{n-1}}\frac{1}{+b_{n-1}}\frac{1}{+z}\right\},$$

where

$$a_i = F(x_{i+1}) - F(x_i), \quad b_i = G(x_{i+1}) - G(x_i),$$
$$x = x_0 < x_1 < \cdots < x_n = t,$$

and, in the indicated limit, $\max(x_{i+1} - x_i) \to 0$. Considering $A, \cdots, D$ as functions of x, the author had found the relations

$$\frac{\partial}{\partial x}A(x,t) = -C(x,t)F'(x), \cdots, \frac{\partial}{\partial x}D(x,t) = -B(x,t)G'(x).$$

He now investigates $A, \cdots, D$ as functions of t and gets the relations

$$\frac{\partial}{\partial t}A(x,t) = B(x,t)G'(t), \cdots, \frac{\partial}{\partial t}D(x,t) = C(x,t)F'(t).$$

The ideas are extended to a continuous continued fraction involving four functions F, G, H, K, with partial quotients $(1+\Delta H_i)/\Delta F_i$, $(1+\Delta K_i)/\Delta G_i$, which reduces to the preceding if H and K are constant. *H. S. Wall.*

Theory of Sets, Theory of Functions of Real Variables

***Natanson, I. P. Theory of functions of a real variable.** Translated by Leo F. Boron with the collaboration of Edwin Hewitt. Frederick Ungar Publishing Co., New York, 1955. 277 pp. $6.50.

Translation of chapters I to IX of the author's Teoriya funkciĭ veščestvennoĭ peremennoĭ [Gostehizdat, Moscow, 1950; MR **12**, 598]. In addition to minor modifications of the text there are also added appendices to chapters III, IV, VI, VII and VIII.

***Huntington, Edward V. The continuum and other types of serial order. With an introduction to Cantor's transfinite numbers. 2d ed.** Dover Publications, Inc., New York, 1955. vii+82 pp. Paperbound $1.00; clothbound $2.75.

Reprint by photo-offset of the 2d edition [Harvard Univ. Press, 1917].

Ginsburg, S. Decompositions of a set into disjoint pairs. Fund. Math. **41**, 278–283 (1955).

The principal result (Th. 1.1) is a generalization of Lemma 3 in Sierpiński, Fund. Math. **24**, 43–47 (1934). Let F be a family of sets of power $\aleph_\gamma$ each; suppose that $(F, \subseteq)$ be coinitial with a subfamily $G = \{B_\xi | \xi < \omega_\gamma\}$. Then there exists a decomposition of $A = \bigcup X$ $(X \in F)$ into disjoint pairs of points so that for any subfamily of these pairs their union S intersects each $X \in F$ in a set of power $\aleph_\gamma$, provided that S contains an element of F (Th. 1.1). In theorems 2.1, 2.2 the author deals with decompositions of the plane R^2 into disjoint pairs. *G. Kurepa* (Zagreb).

Popruženko, J. Sur une propriété des transformations des ensembles abstraits. Fund. Math. **41**, 163–167 (1955).

Let $|Y| \geq |X| > \aleph_0$. Suppose that μ^* is a finite exterior measure defined for the subsets of X, not vanishing identically, but vanishing for every element of X. Denote by $\mathfrak{a}$ the smallest aleph for which there exists a family, of power $\mathfrak{a}$, of subsets of X of μ-measure 0, whose union U satisfies the relation $\mu^*(U) > 0$. Then $\mathfrak{a}$ is a regular aleph with $\aleph_1 \leqq \mathfrak{a} \leqq |X|$. Let Φ be a family, of power $\mathfrak{a}$, of subsets of Y of power $\geqq \mathfrak{a}$, and Ψ be a family, of power $\mathfrak{a}$, of single-valued functions $y = f(x) \in Y$ $(x \in X)$ such that, if $f \in \Psi$ and $y \in f(X)$, then $\{x: x \in X, f(x) = y\}$ is μ-measurable. Then, for every $M \subset X$ with $\mu^*(M) > 0$, there exists an $N \subset M$ with $|N| = \mathfrak{a}$, such that, if $E \in \Phi$ and $f \in \Psi$, $E - f(N) \neq 0$. This generalizes results of Sierpiński [Fund. Math. **19**, 205–210 (1932)]. *F. Bagemihl* (Princeton, N. J.).

Ellis, David. On infinite series of sets. Proc. Glasgow Math. Assoc. **2**, 89–92 (1954).

Proof of a fact concerning sums of sets. The author fails to notice that it is contained in the known case of a group with certain subgroups as basic neighborhoods of the identity. *L. Nachbin* (Rio de Janeiro).

Botts, Truman. On lattice embeddings for partially ordered sets. Canad. J. Math. **6**, 525–528 (1954).

Proof of a known fact concerning the imbedding of a partially ordered set in the complete lattice of its increasing subsets, with a reference to lattices of topologies.

L. Nachbin (Rio de Janeiro).

Salem, R. Uniform distribution and capacity of sets. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 193–195 (1952).

Consider the infinite sequence $\{n_k x\}$, where x is a real number between 0 and 1 and the n_k are distinct positive integers. It has been shown by H. Weyl [Math. Ann. **77**, 313–352 (1916)] that the terms of this sequence are uniformly distributed modulo 1, in the sense that the ratio

$$\rho(k) = \frac{1}{k}(e^{2\pi i n_1 x} + \cdots + e^{2\pi i n_k x})$$

tends to zero as $k\to\infty$ for every positive integer h, for almost all values of x between 0 and 1 in the sense of Lebesgue measure. Under the additional assumption that $\max(n_1, n_2, \cdots, n_k)=O(k^p)$ for some $p\geqq 1$, the author shows that Lebesgue measure can be replaced by the smaller "measure" of capacity with respect to a generalized potential $\int r^{-\alpha}d\mu(x)$, usually referred to as α-capacity, for any $\alpha>1-p^{-1}$.

It is also noted that some restriction of the above type on the rate of growth of the n_k is necessary for these sharpened results. This is proved by the construction of a sequence $\{n_k\}$, for any two numbers $c>1$ and $\alpha<1$, such that $\max(n_1, \cdots, n_k)=O(c^k)$ and such that the exceptional set of x between 0 and 1 for which the terms of $\{n_kx\}$ are not uniformly distributed modulo 1 has positive α-capacity.

B. Lepson (Washington, D. C.).

Haupt, O., and Pauc, Chr. Propriétés de mesurabilité de bases de dérivation. Portugal. Math. **13**, 37–54 (1954).

Sufficient conditions for bases of derivation with measurable domain and measurable extreme derivates for every finite function are given by the authors in Haupt-Aumann-Pauc, Differential- und Integralrechnung, Bd. III, pp. 220–223 [2d ed., Berlin, 1955]. The investigations of the present paper are even further reaching. Fundamental is the notion of the "prébase". Let R be the fundamental set, let G be a set of directed Moore-Smith sequences g of sets of R, and let a be a relation between the points of R and the sequences g, i.e. a subset of $R\times G$. Then a is called a "prébase" if it satisfies the following "axiom of heredity": every confinal subsequence of a sequence g which converges to a point x belongs to G and converges to x also. The authors call "constituent" every element of any g and they call a finite real-valued set function m, defined on the family $\mathfrak{v}$ of all constituents, a "base function". Then a "prébase" bearing a base function is a "general base of derivation". The derivates of a set function F defined on $\mathfrak{v}$ are formed with respect to m. It is remarkable that the sufficient conditions given by the authors for the measurability of the domain (§2) and of the extreme derivates of every F (§3) refer only to the "prébase" and not to the base function m. In §4 examples satisfying those conditions are given and, in particular, the bases of derivation of A. Denjoy [Amer. J. Math. **73**, 314–356 (1951); MR **12**, 685] are taken into consideration. Finally, in §5 a more complicated application is given: a parameter of regularity (in the sense of Lebesgue) is introduced and the measurability of "regular extensions" of bases of derivation is discussed. *A. Rosenthal.*

Lehman, R. Sherman. Approximation of improper integrals by sums over multiples of irrational numbers. Pacific J. Math. **5**, 93–102 (1955).

Let $f(x)$ be periodic mod 1, improperly Riemann integrable over $(0, 1)$, bounded in every closed sub-interval, and near the end-points 0 and 1 of the interval $(0, 1)$ either bounded or monotone. Let α be an irrational number, and $q_1, q_2, \cdots$ the denominators of its convergents.

The author proves the following results.

$$(1)\qquad \lim_{N\to\infty} N^{-1}\sum_{1\leq n\leq N}{}^{*} f(n\alpha)=\int_0^1 f(x)dx,$$

where the * indicates the omission of those n which are multiples of q, where q is the largest q_i not exceeding N. (2) If the quotients in the continued fraction for α are bounded, then the asterisk in (1) can be omitted. (3) The asterisk can be omitted for almost all α. (4) Formula (1) can be extended to the case that $f(x)$ has a finite number of singularities (of the monotone type) other than 0 or 1.

These results are extensions of Weyl's well-known result [Math. Ann. **77**, 313–352 (1916)] that (1) holds without asterisk, if $f(x)$ is properly integrable. *N. G. de Bruijn.*

Leader, Solomon. On universally integrable functions. Proc. Amer. Math. Soc. **6**, 232–234 (1955).

Let A be a Boolean algebra of subsets of a set X. A real-valued function f on X is called universally integrable if the sums $\sum f(x_k)F(E_k)$ tend to a finite Moore-Smith limit whenever F is a real-valued, finitely additive measure on A; here $\{E_k\}$ is a finite partition of X consisting of sets in A, and x_k is in E_k for each k. A step-function is a finite linear combination of characteristic functions of sets in A. The author proves that a function is universally integrable if and only if it is a uniform limit of step functions.

P. R. Halmos (Chicago, Ill.).

Šmidov, F. I. On the theory of the integral. Dokl. Akad. Nauk SSSR (N.S.) **101**, 31–34 (1955). (Russian)

The author generalizes a theorem giving sufficient conditions that a real-valued function of a real variable be of generalized bounded variation. Let F be finite-valued on a bounded set E of real numbers, and call a point x a point of positive density for F if for each $r>0$ the set

$$\{y\,|\,|F(x)-F(y)|<r \text{ and } |x-y|<r\}$$

has x as a point of positive lower density. Theorem 1. If E' is the set of points of positive density for F, and if at each point of E', with the possible exception of a countable set E'', the function F has at least one finite approximate derivative number, then F is of generalized bounded variation in the wide sense on E' [Saks, Theory of the integral, 2nd ed., Warszawa-Lwów, 1937, has the pertinent definitions and some less general results. See Chapter VII, §10 and Chapter IX, §9.] *M. M. Day* (Urbana, Ill.).

Ionescu Tulcea, C. T. Fonctions d'ensemble et leurs intégrales. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 73–142 (1954). (Romanian. Russian and French summaries)

This paper contains a systematic treatment of very general notions of integration. The integrals are defined for certain functions f defined on an object called a groupoid and whose values are sets in a semi-group. Here a semi-group is a set with an operation "+" which is commutative, associated, and possesses an identity element. The semi-groups are assumed to possess a uniform structure in which the operation "+" is uniformly continuous in both variables. An example is a locally convex linear topological space under addition. A groupoid is defined to be a set G with an operation "$\circ$" which is defined only for certain pairs $x, y\in G$ but otherwise satisfies the same conditions as the operation in a semi-group. An example of a groupoid is the set of all subdivisions of a set E where elements of a subdivision belong to a special class T of subsets of E. Here a subdivision is any finite collection of disjoint elements of T. The operation $x\circ y$ is defined for two subdivisions provided every set involved in x is disjoint from every set in y, in which case $x\circ y$ is the subdivision consisting of all sets in x and y together. The author considers integrals both of the Riemann and the Lebesgue type. In the latter case the groupoid is assumed to be "complete" in the sense that its operation is defined for certain countable sets. The moti-

vating example here is the same as the one given above except subdivisions are allowed to involve a countable infinity of disjoint sets. The various integrals are defined using certain filters in the groupoid G. The integrals obtained include in one way or another integrals studied by Kolmogoroff [Math. Ann. **103**, 654–696 (1930)], Phillips [Trans. Amer. Math. Soc. **47**, 114–145 (1940); MR **2**, 103], Rickart [ibid. **52**, 498–521 (1942); MR **4**, 162], Nikodym [C. R. Acad. Sci. Paris **225**, 479–481 (1947); MR **9**, 83], Cotlar and Frenkel [Univ. Nac. Tucuman. Rev. Ser. A. **6**, 113–159 (1947); MR **10**, 107], Pettis [Trans. Amer. Math. Soc. **44**, 277–304 (1938)], Gelfand [Zap. Nauk.-Doslid. Inst. Mat. Meh. ta Harkiv. Mat. Tow. (4) **13**, 35–40 (1937)], and others. *C. E. Rickart* (New Haven, Conn.).

Marstrand, J. M. Circular density of plane sets. J. London Math. Soc. **30**, 238–246 (1955).

The author sharpens slightly the result of a previous paper [Proc. London Math. Soc. (3) **4**, 257–302 (1954); MR **16**, 121] according to which, for non-integral s, an s-set cannot be regular, by showing that, at almost every point of such a set, the circular density cannot exist. *L. C. Young.*

Delange, Hubert. Sur deux questions posées par M. Karamata. Acad. Serbe Sci. Publ. Inst. Math. **7**, 69–80 (1954).

Karamata's two questions, concerning a real- or complex-valued function f defined for all real $x \geqq$ some x_0, are these. (1) If for each $t>0$ the difference $f(x+t)-f(x)$ is bounded for $x \geqq x_0$, must this, for each $h>0$, hold uniformly in t for $0 \leqq t \leqq h$? (2) If f is real-valued and for each $t>0$ the difference $f(x+t)-f(x)$ is bounded above for $x \geqq x_0$, must this, for each $h>0$, hold uniformly in t for $0 \leqq t \leqq h$? (1) is answered affirmatively in case f is continuous except for simple jump discontinuities x at which $f(x)$ lies on the closed segment with endpoints $f(x+0)$ and $f(x-0)$. An example shows the answer to (2) is negative even for continuous functions, but a related conclusion is shown to follow from the hypothesis of (2). *T. A. Botts* (Charlottesville, Va.).

Marcus, S. Les fonctions de Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 413–419 (1954). (Romanian. Russian and French summaries)

Let a_n be a sequence of points dense on the interval (x', x''). Then the function $t=F(x)=\sum_{n=0}^{\infty} A_n(x-s_n)^{1/3}$ with $A_n>0$, and $\sum_n A_n<\infty$, is continuous, monotonic nondecreasing, has an infinite derivative where $\sum_n A_n(x-a_n)^{-2/3}$ diverges or $x=a_n$, and a finite derivative elsewhere. Moreover, the function $F'(x)$ is bounded from zero. The inverse function $x=G(t)$ on the interval (t', t'') is the function of Pompeiu [Math. Ann. **63**, 326–332 (1907), Pompeiu assumes $\sum_n \sqrt{A_n}<\infty$] and has a derivative at every point of (t', t''); the derivative function is bounded and assumes the value 0 at a set of points dense on (t', t''), and so is not Riemann integrable. Its graph is useful in the construction of pathological spaces. [See, e.g., Knaster and Kuratowski. Rend. Circ. Mat. Palermo **49**, 382–386 (1926), p. 385.]

T. H. Hildebrandt (Ann Arbor, Mich.).

Marcus, S. Propriétés bidimensionnelles, métriques et qualitatives, des fontions réelles de deux variables réelles. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 527–544 (1953). (Romanian. Russian and French summaries)

The author terms bidimensional limit of $f(x, y)$ as $(x, y)\to(a, b)$ the corresponding limit of the expression $f(x, b)+f(a, y)-f(x, y)$. He then considers the effect of substituting this notion of limit in various definitions of continuity. The appropriate theorems and the lemmas used in their proofs are stated at length in the summaries.

L. C. Young (Madison, Wis.).

Marcus, S. La composition des fonctions à variation bornée. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 243–250 (1954). (Romanian. Russian and French summaries)

The author's main result is as follows: Let $g(x)$ be the superposition $f[h(x)]$ of two functions of bounded variation on appropriate linear intervals. Then the set of the points of non-derivability of $g(x)$ is the sum of three sets $U+V+W$, where $|U|=|h(V)|=|g(W)|=0$ and where V has a characterization independent of the functions f and h. The author treats also various cases in which f and h satisfy further conditions but are not both absolutely continuous.

L. C. Young (Madison, Wis.).

Haršiladze, F. I. Functions with bounded second variation. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 145–156 (1954). (Russian)

A function $f(x)$ defined and measurable in $[a, b]$ is said to have a bounded second variation, if for any subdivision $a=x_0<x_1<\cdots<x_n=b$ the sum $\sum_{k=0}^{n-1}|\Delta^2(f; x_k, x_{k+1})|$, where $\Delta^2(f, \alpha, \beta)=f(\alpha)+f(\beta)-2f((\alpha+\beta)/2)$, remains bounded. It is shown that such functions can have at most a denumerable set of discontinuities, all of the first kind. If $[a, b]=(0, 2\pi)$, and a_n, b_n are the Fourier coefficients of f then, with $\rho_n=(a_n^2+b_n^2)^{1/2}$, each of the conditions

$$\liminf n^{-2}\sum_1^n \rho_k k^2=0, \quad \liminf (\log n)^{-1}\sum_1^n \rho_k=0$$

is both necessary and sufficient for the continuity of f.

A. Zygmund (Chicago, Ill.).

Amanov, T. I. Boundary functions of classes $H_p^{(r_1,\cdots,r_n)}$ and $H_p^{*(r_1,\cdots,r_n)}$.

Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 17–32 (1955). (Russian)

The classes of the title are generalized Lipschitz classes involving higher derivatives of functions of several variables; they were introduced and studied by Nikolskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **76**, 785–788 (1951); Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); MR **12**, 603; **14**, 32]. The author constructs examples to show that some results of Nikolskiĭ cannot be improved [cf. his preliminary announcement, Dokl. Akad. Nauk SSSR (N.S.) **88**, 5–8 (1953); MR **15**, 205]. *R. P. Boas, Jr.*

Nevanlinna, Rolf. Über die Umkehrung differenzierbarer Abbildungen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **185**, 12 pp. (1955).

The author considers a mapping $x\to y(x)$ between two real unitary spaces of equal finite dimension. He obtains an inversion theorem under the assumption that the derivative y' exists in a neighborhood of $x=0$ where it is nonsingular, assuming only that y' is continuous at $x=0$ (instead of in a neighborhood of this point). An implicit function theorem is obtained under similar hypotheses. It is asserted that the results remain valid for infinite-dimensional unitary spaces. The results are less extensive than those obtained in Banach spaces by T. H. Hildebrandt and L. M.

Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927)], but the hypotheses of the present paper are less restrictive in that the functions are not necessarily in class C'.

R. G. Bartle (Urbana, Ill.).

Krickeberg, Klaus. Über den Gaussschen und den Stokesschen Integralsatz. III. Math. Nachr. **12**, 341–365 (1954).

[For parts I and II see Math. Nachr. **10**, 261–314 (1953); **11**, 35–60 (1954); MR **15**, 611, 692.] In this expository article there is given a comprehensive and critical account of the proofs of Gauss's and Stokes's theorems that have been published during the last 150 years. The paper concludes with a long list of references to the literature.

H. G. Eggleston (Cambridge, England).

Theory of Functions of Complex Variables

*Heffter, Lothar. **Begründung der Funktionentheorie auf alten und neuen Wegen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+63 pp. DM 15.60.

With many historical references, the author presents several equivalent definitions of analyticity in complex variable theory, including those associated with the names of Weierstrass, Cauchy-Goursat, Looman-Menchoff, Morera-Osgood, and Heffter. *E. F. Beckenbach.*

Menger, Karl. A simple definition of analytic functions and general multifunctions. Proc. Nat. Acad. Sci. U. S. A. **40**, 819–821 (1954).

The author makes precise the somewhat vague notion of a multiple-valued analytic function of a complex variable by defining an analytic multifunction to be a class F of ordered pairs of complex numbers such that (1) each pair of F belongs to a single-valued analytic function which is a subclass of F, (2) corresponding to each pair f_0 and f_1 of single-valued analytic functions which are subclasses of F there exists a "motion" $(\xi, \eta)[0, 1]$ such that for some $\epsilon>0$ segmental motions $(\xi, \eta)[0, \epsilon]$ and $(\xi, \eta)[1-\epsilon, 1]$ belong to f_0 and f_1 respectively, and if $0<t<1$, then for some $\epsilon>0$ the motion $(\xi, \eta)[t-\epsilon, t+\epsilon]$ belongs to some single-valued analytic function which is a subclass of F. By a motion $(\xi, \eta)[t_0, t_1]$ is meant an ordered pair $\xi(t)$, $\eta(t)$ of continuous complex-valued functions of the real variable t, defined for $t_0 \leqq t \leqq t_1$.

He defines branch points and homotopy groups of an analytic multifunction. He also defines general multifunctions, continuous multifunctions, and polyfunctions, and from the point of view of tri-operational algebra considers the operation of substituting one multifunction in another.

O. Frink (University Park, Pa.).

Capyrin, V. N. The Routh-Hurwitz problem for a quasi-polynomial for $s=1$, $r=5$. Inžen. Sb. **15**, 201–206 (1953). (Russian)

The Routh-Hurwitz problem for an analytic function is the determination of the necessary and sufficient conditions that all the zeroes have negative real parts. This is a stability condition of importance in the theory of differential, differential-difference and other types of functional equations. Using the classical approach of Cauchy and Hermite, a method was developed by Pontrjagin [Izv. Akad. Nauk SSSR. Ser. Mat. **6**, 115–134 (1942); MR **4**, 214], following Chebotarev, to treat analytic functions having the form $h(z, e^z)=0$, where $h(x, y)$ is a polynomial in x and y. The author treats the case where

$$h(z, e^z) = (a_0+a_1z+\cdots+a_kz^k)\cosh z + (b_0+b_1z+\cdots+b_kz^k)\sinh z$$

using an elementary technique. *R. Bellman.*

Kantz, Georg. Beziehungen zwischen den Koeffizienten einer analytischen Funktion und ihrer Umkehrfunktion. Monatsh. Math. **59**, 27–33 (1955).

Let $w=f(z)$ be regular at z_1, with $w_1=f(z_1)$ and $f'(z_1)\neq 0$; and let $z=\phi(w)$ be its inverse function, the latter being regular at w_1. The author derives explicit formulas for the derivatives $\phi_n=\phi^{(n)}(w_1)$, $n=1, 2, \cdots$, in terms of the derivatives $f_m=f^{(m)}(z_1)$, $m=1, 2, \cdots$. More precisely, ϕ_n is shown to be a polynomial in $f_1^{-1}, f_2, f_3, \cdots, f_n$; the coefficients of these polynomials are integers which are expressed in terms of factorials. *F. Herzog* (East Lansing, Mich.).

Gabriel, Richard F. The Schwarzian derivative and convex functions. Proc. Amer. Math. Soc. **6**, 58–66 (1955).

The principal result is that if $f(z)=z^{-1}+a_1z+a_2z^2+\cdots$ for $0<|z|<1$, and if the Schwarzian derivative $\{f(z), z\}$ satisfies $|\{f(z), z\}|\leqq 2c^2$, where c is the smallest root of $\tan x=2x$, then $f(z)$ is univalent in $0<|z|<1$ and maps $|z|<r<1$ into the exterior of a convex region. The constant c is best possible. The methods used involve the representation of $f(z)$ as the quotient of two solutions of the differential equation $w''+p(z)w=0$. The constant $2c^2$ is smaller than Z. Nehari's $\frac{1}{2}\pi^2$ [Bull. Amer. Math. Soc. **55**, 545–551 (1949); MR **10**, 696] when the conclusion is merely that $f(z)$ is univalent in $|z|<1$. *A. Macintyre* (Aberdeen).

Karcivadze, I. N., and Hvedelidze, B. V. On an integral of Cauchy type. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 211–244 (1954). (Georgian. Russian summary)

Let C_k $(k=1, 2, \cdots)$ be a denumerable set of smooth open or closed curves; $C=\sum C_k$; $\int_C=\sum \int_{C_k}$. Class B (or B^*) is the set of $\phi(t)$ such that: (1°) $\phi(t) \in I_\infty(C_k)$ $(k=1, \cdots)$ [the definition of I_∞ is due to L. G. Magnaradze, Soobšč. Akad. Nauk Gruzin. SSR **8**, 509–516 (1947); MR **14**, 152]; (2°) the series $\int_C \phi(t)(t-z)^{-1}dt$ converges uniformly (or uniformly and absolutely) in every domain at a positive distance from C. A chain is any set of curves $C_{k_1}, C_{k_2}, \cdots, C_{k_p}$ (p finite), with $C_{k_i}C_{k_{i+1}}\neq 0$ $(i=1, \cdots, p-1)$; C_k^* is the set of all C_j joinable with C_k by a chain; a ξ-point of C is any point of accumulation of an infinity of different C_k; C is n. (normal) if: $(1_\circ)$ every C_k^* consists of a finite number of C_j; $(2_\circ)$ no ξ-point of C is on C. Some of the results are as follows. If C is n. and $\phi(t) \in B$, then

$$2\pi i\Phi(z)=\sum\Phi_k(z)=\sum\int_{C_k}\phi(t)(t-z)^{-1}dt$$

is uniformly convergent on every closed bounded set devoid of ξ-points. $\Phi(z)$ converges on every C_k, if C is n. and $\phi(t) \in B$; if, moreover, $C_kC_j=0$ $(k\neq j)$, then $\phi(t) \in I_\infty(C_k)$ on C_k. C is termed i. (isolated) if: (1) no ξ-point is on C; (2) the C_i are disjoint; (3) some C_j is a closed curve and $D_jC=0$, where D_j is the domain interior to C_j. Plemelj (Sohotsky) formulas hold if C is i. If C is i., its components being closed curves, and $\phi(t) \in B$, then $\int_C(t-x)^{-1}dt\int_C\varphi(\tau)(\tau-t)^{-1}d\tau=-\pi^2\phi(x)$ on C; this formula still holds when $\phi(t) \in A$, where A is the class of all $\phi(t)$ such that: $\phi(t) \in I_\infty(C_k)$ $(k=1, 2, \cdots)$, $\int_C\varphi(t)(t-x)^{-1}dt$ converges uniformly on each C_k and the

sum of this series is in $I_\infty(C_k)$ on each C_k. The above results are applicable in a natural way to corresponding singular integral equations with kernels of Cauchy type.

W. J. Trjitzinsky (Urbana, Ill.).

Biernacki, Mieczysław. Sur quelques applications de la formule de Parseval. II. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 5–14 (1954). (Polish and Russian summaries)

[For part I, see same Ann. **4**, 23–40 (1950); MR **13**, 123.] Let the functions $f(z)=\sum a_n z^n$ and $g(z)=\sum b_n z^n$ be regular in $|z|<1$, let $H(f,g,x)=H(x)=\sum a_n b_n x^n$, $M(r)=\max_{|z|=r}|f(z)|$ $(0<r<1)$,

$$I_k(r,f)=\left[(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^k d\theta\right]^{1/k} \quad (k>0),$$

and let $S(r,f)$ denote the area of the Riemann surface upon which $f(z)$ maps the disc $|z|<r$. The author uses the formula

$$H(f,g,x)=(2\pi i)^{-1}\int_C f(z)g(x/z)z^{-1}dz,$$

where C is the circle $|z|=|x|^{1/2}$. With the notation $\rho=r^{1/2}$, the basic inequalities which he obtains become

$$M(r,H)\leqq I_1(\rho,f)M(\rho,g),$$
$$I_1(r,H)\leqq I_1(\rho,f)I_1(\rho,g),$$
$$M(r,H)\leqq I_p(\rho,f)I_q(\rho,g) \quad (p^{-1}+q^{-1}=1).$$

Since the function $\sum n|a_n|^2x^{2n}$ is the H-composition of the functions $\sum na_nz^{2n}$ and $\sum \bar{a}_nz^{2n}$, it follows, for example, that

$$S(r,f)\leqq \pi r M(r,f)I_1(r,f'),$$

and therefore that $S(r,f)\leqq RL/2$, where L is the length of the boundary of the Riemann surface upon which $f(z)$ maps the disc $|z|<r$, and R is the radius of the least circular disc that covers this surface. If $f(z)\neq 0$ in $|z|<1$, the replacement of f by zf', together with the special choice $g=\sum \bar{a}_nz^n$, leads to the results

$$S(r,f)\leqq \pi M(r,zf'/f)I_2^2(r,f),$$
$$S(r,f)\leqq \pi I_1(r,zf'/f)M^2(r,f).$$

The author also applies his technique to functions which are p-valent in the mean [see Spencer, Proc. London Math. Soc. (2) **47**, 201–211 (1941); MR **3**, 79], and to the study of the mean square modulus. Here he obtains, for example, the inequality

$$I_2^2(r,f')\leqq M\left(r,\frac{f-a_0}{z}\right)I_1(r,f'+zf'').$$

G. Piranian (Ann Arbor, Mich.).

Repin, I. I. On sequences of uniformly bounded growth of linear aggregates of analytic functions. Mat. Sb. N.S. **36(73)**, 3–24 (1955). (Russian)

The author considers sequences of the form

$$\psi_n(z)=\sum_{j=1}^{n}\gamma_{n,j}f(\lambda_j z),$$

where $f(z)$ is an entire function of order ρ, $\{\lambda_n\}$ is a sequence whose exponent of convergence is $\rho_1>\rho$, and $\psi_n(z)$ converges over the finite plane and satisfies $|\psi_n(z)|<\exp(|z|^\omega)$, $|z|>r_0$, with $\omega^{-1}>\rho^{-1}-\rho_1^{-1}$. His object is to deduce further properties of the sequence: for example, $\lim_{n\to\infty}\gamma_{n,j}=\gamma_j$ and the γ_j depend only on the function $\lim\psi_n(z)$; and under some auxiliary conditions the series $\sum\gamma_j f(\lambda_j z)$ actually converges to $\lim\psi_n(z)$. The proofs depend on Leont'ev's M-operators [Mat. Sb. N.S. **33(75)**, 453–462 (1953); MR **15**, 301], whose theory is developed more fully by the author.

R. P. Boas, Jr. (Evanston, Ill.).

Hiong, King-Lai. Sur certaines valeurs exceptionnelles des fonctions holomorphes dans le cercle-unité et de leurs dérivées. C. R. Acad. Sci. Paris **240**, 1685–1687 (1955).

The author continues his work [Ann. Sci. Ecole Norm. Sup. (3) **70**, 149–180 (1953); MR **15**, 412] on Schottky-type inequalities for functions $f(z)$ analytic in $|z|<1$ for which 0 and 1 are exceptional values of some kind. In the present paper f has at most p zeros in $|z|<1$, and either f or its kth derivative has 1 as B-exceptional value. Here α is called B-exceptional for the function $g(z)$ of supposedly unbounded characteristic $T(r)$ when

$$\limsup_{r\to 1} N(r,(g-\alpha)^{-1})/\log(1-r)^{-1}<\infty.$$

J. Korevaar (Madison, Wis.).

Edrei, Albert. Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc. **78**, 276–293 (1955).

Let $f(z)$ be meromorphic in the z-plane. The roots of the equation $f(z)=a$ are said to be distributed on the q radii (1) $re^{i\omega_s}$ $(s=1,2,\cdots,q;\ r\geqq 0)$, if all but a finite number of these roots lie on the given radii. The author's main result is concerned with the order ρ of $f(z)$. Let the roots of the three equations

$$f(z)=0,\quad f(z)=\infty,\quad f^{(l)}(z)=1 \quad (l\geqq 0,\ f^{(0)}\equiv f)$$

be distributed on the radii (1). Denote by $\delta(a,f^{(l)})$ the deficiency of the value a, of $f^{(l)}$, and assume

$$\delta(0,f)+\delta(1,f^{(l)})+\delta(\infty,f)>0. \tag{2}$$

Then ρ is finite and

$$\rho\leqq\beta=\sup\left\{\frac{\pi}{\omega_2-\omega_1},\frac{\pi}{\omega_3-\omega_2},\cdots,\frac{\pi}{\omega_{q+1}-\omega_q}\right\},\quad \omega_{q+1}=2\pi+\omega_1.$$

If (2) is replaced by the condition that the roots of the three given equations have a finite exponent of convergence ρ_1, then again ρ is finite and either $\rho=\rho_1$, or else $\rho_1<\rho\leqq\beta$. In particular, if $f(z)$ is an entire function all of whose zeros are real, and if the roots of $f^{(l)}(z)=1$ $(f^{(0)}\equiv f)$ are all real for one integer $l\geqq 0$, then the order of $f(z)$ is finite and does not exceed one ($\frac{1}{2}$ if the word "real" is replaced by the word "positive"). *M. S. Robertson* (New Brunswick, N. J.).

Clunie, J. On a theorem of Collingwood and Valiron. J. London Math. Soc. **30**, 228–231 (1955).

Let $f(z)$ be regular in a simply connected domain D with a boundary E consisting of a simple, closed, continuous curve passing through the origin 0. Let $f(z)$ have an isolated essential singularity at the origin on E, and be bounded on E, continuous on E except possibly at 0. Let

$$m(r)=\max_{|z|\leq r,\ z\in E}|f(z)|,\quad M(r)=\max_{|z|=r,\ z\in D}|f(z)|.$$

It is supposed that $M(r)\to\infty$ as $r\to 0$. Let R_0 be the upper bound of r for which $M(r)$ is an increasing function of $1/r$ and such that $|f(z)|=M(r)$ for a point z on an arc of $|z|=r$ which partly bounds a sub-domain of D, the remainder of whose boundary is part of E passing through 0. The author gives a new proof of a theorem of Collingwood and Valiron [Proc. London Math. Soc. (2) **26**, 169–184 (1927)] to the effect that there exists an absolute constant $k<1$, such that, for $r<R_0$, $M(kr)m(r)>M^2(r)$. By means of the new proof

he obtains $k \geqq e^{-4\pi}$ which is an improvement on the bound for k obtained by the above authors. *M. S. Robertson.*

Clunie, J. Univalent regions of integral functions. Quart. J. Math., Oxford Ser. (2) **5**, 291–296 (1954).

Let $f(z)$ be an entire function of positive order less than $\frac{1}{2}$. The author shows that there are an infinite sequence of simply connected domains D_q, with minimum distance from the origin tending to ∞, and numbers A_q with $A_q \to \infty$, such that each D_q is mapped by $w=f(z)$ univalently either onto $|w|<A_q$ or onto this circle minus at most $2e(2-\log\cos\pi\rho)\sec\pi\rho$ segments extending into its interior.

R. P. Boas, Jr. (Evanston, Ill.).

Linis, Viktors. Note on univalent functions. Amer. Math. Monthly **62**, 109–110 (1955).

A short proof of a theorem by B. Friedman [Duke Math. J. **13**, 171–177 (1946); MR **8**, 22] on power series, univalent in the unit circle, and with integral coefficients.

W. W. Rogosinski (Newcastle-upon-Tyne).

Iliev, Lyubomir. On the difference quotient for bounded univalent functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 861–862 (1955). (Russian)

Bazilevič proved [Mat. Sb. N.S. **28(70)**, 283–292 (1951); MR **13**, 640] that if $f(z)$ is regular, univalent and $|f(z)|<M$ in $|z|<1$, then

$$\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \geqq \prod_{i=1}^{2}\left|\frac{f(z_i)}{z_i}\right|\left(\frac{1-|z_i|^2}{1-|f(z_i)|^2/M^2}\right)^{1/2}.$$

The author announces a generalization of this type of inequality to the case where in addition the image domain under $f(z)$ has k-fold symmetry. *A. W. Goodman.*

Iliev, Lyubomir. Theorem on the univalence of finite sums of triply symmetric univalent functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 621–622 (1955). (Russian)

Let $f(z)=z+\sum_{n=1}^{\infty} a_n z^{3n+1}$ be regular and univalent in $|z|<1$. Then the sum of the first n terms is univalent in $|z|<3^{1/3}/2$, and this constant cannot be decreased.

A. W. Goodman (Lexington, Ky.).

Tsuji, Masatsugu. On the radial order of a certain regular function in a unit circle. J. Math. Soc. Japan **6**, 336–342 (1954).

The author gives a simple proof of the known theorem that, if $f(z)$ is regular and univalent in $|z|<1$, there exists a set E of measure 0 on $|z|=1$ such that for every $e^{i\theta}$ non-ε E, $f'(z)=o(|z-e^{i\theta}|^{-1/2})$ uniformly as $z\to e^{i\theta}$ within any Stolz angle with vertex $e^{i\theta}$ [W. Seidel and J. L. Walsh, Trans. Amer. Math. Soc. **52**, 128–216 (1942), pp. 141–146; MR **4**, 215]. The following related result is also proved: Let $w=f(z)$ be regular in $|z|<1$ and suppose that for some $p>0$

$$\iint_{|z|<1} |f(z)|^p dxdy < \infty \quad (z=x+iy).$$

Then there exists a set E of measure 0 on $|z|=1$ such that, for every $e^{i\theta}$ non-ε E, (i) if p is a positive integer, $f(z)=O(|z-e^{i\theta}|^{-1/p})$; (ii) if p is not a positive integer, for any $\delta>0$, $f(z)=O(|z-e^{i\theta}|^{-(1+\delta)/p})$ uniformly as $z\to e^{i\theta}$ within any Stolz angle with vertex $e^{i\theta}$. *W. Seidel.*

***Heins, Maurice. The set of asymptotic values of an entire function.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 56–60 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

D'après Mazurkiewicz [Fund. Math. **17**, 26–29 (1931)] l'ensemble des valeurs asymptotiques d'une fonction entière est un ensemble analytique contenant le point à l'infini [en bref: ensemble A]. Inversement, étant donne un ensemble A, l'auteur construit une fonction entière l'admettant pour ensemble des valeurs asymptotiques.

J. P. Kahane (Montpellier).

Andreian, Cabiria. Sur la théorie de R. Nevanlinna. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 271–296 (1954). (Romanian. Russian and French summaries)

This paper is concerned with developing the Nevanlinna theory for meromorphic functions in the finite plane with the modification that the exhausting regions instead of being taken as concentric circular disks are taken to be the members of a monotone 1-parameter family of simply-connected regions with analytic boundary, the family being restricted to satisfy certain conditions of continuity. Theorems generalizing those of the Nevanlinna theory are established. It is shown by example that the notion of an exceptional value is relative and depends upon the exhaustion.

M. Heins (Providence, R. I.).

Kennedy, P. B. On a conjecture of Heins. Proc. London Math. Soc. (3) **5**, 22–47 (1955).

This paper refines the results of the reviewer [Ann. of Math. (2) **49**, 533–537 (1948); MR **10**, 28] concerning the Denjoy-Carleman-Ahlfors theorem. Let $C_1, C_2, \cdots, C_n$ denote n Jordan arcs in the extended plane, each joining $z=0$ to $z=\infty$ and such that for $k\neq l$, C_k and C_l have only $z=0$ and $z=\infty$ in common. It is assumed that the indexing is such that the n regions $D_1, \cdots, D_n$ which are the components of the complement of $\bigcup_1^n C_k$ have as their respective frontiers $C_1\cup C_2, \cdots, C_n\cup C_{n+1}$ ($C_{n+1}=C_1$). Let u_k denote a subharmonic function with domain D_k which is not bounded above and satisfies $\limsup u_k \leqq 0$ at each finite frontier point of D_k. Let $\sigma_k(r)=\sup_{|z|=r, z\in D_k} u_k(z)$ and let $\sigma(r)=\max \sigma_k(r)$. The reviewer showed [loc. cit.] that, if $\liminf_{r\to\infty} r^{-n/2}\sigma(r)<+\infty$, then $\lim_{r\to\infty}\log\sigma_k(r)/\log r=\frac{1}{2}n$ ($k=1, \cdots, n$) and

$$0<\liminf r^{-n^2/2}\prod_1^n \sigma_k(r) \leqq \limsup r^{-n^2/2}\prod_1^n \sigma_k(r)<+\infty$$

The author shows that under this hypothesis:

(a) $\log\sigma_k(r)-\frac{1}{2}n\log r=o[(\log r)^{1/2}]$ $(k=1, \cdots, n)$,

(b) $$\prod_1^n \sigma_k(r)\sim\gamma r^{n^2/2},$$

where γ is a finite positive constant. The methods of the author use results of W. K. Hayman [Quart. J. Math., Oxford Ser. **19**, 33–53 (1948); MR **9** 420; and a lemma (Lemma 4) given in a lecture]. Results pertaining to the Ahlfors rotation index are also established. In the reviewer's paper it was stated that it was highly plausible that the hypothesis $\liminf r^{-n/2}\sigma(r)<+\infty$ implied that $\lim r^{-n/2}\sigma(r)$ existed and it was noted that this assertion was true for $n=1$ [M. Heins, Ann. of Math. (2) **49**, 200–213 (1948); MR **9**, 341]. The author notes that this is not the case for $n>1$ and that in a forthcoming paper a construction of a

class of entire functions will be given which shows that his results are best possible. *M. Heins* (Providence, R. I.).

Rudin, Walter. On a problem of Bloch and Nevanlinna. Proc. Amer. Math. Soc. **6**, 202–204 (1955).

The author constructs a power series which converges absolutely on the closed unit disc and whose derivative approaches ∞ along almost every radius of the unit circle. The function, represented by this power series inside the unit circle, thus serves as an example of a regular function which is bounded on $|z|<1$ but whose derivative is not of bounded characteristic. The question of the existence of such a function was raised by Bloch, and an example was given by O. Frostman [Kungl. Fysiogr. Sällsk. i Lund Förh. **12**, no. 15, 169–182 (1942); MR **6**, 262]. Unlike Frostman's example, which has the form of an infinite Blaschke product, the function constructed by the present author has the additional property of being continuous on the boundary of the unit circle. *F. Herzog.*

Rudin, Walter. Analytic functions of class H_p. Trans. Amer. Math. Soc. **78**, 46–66 (1955).

L'A. développe pour un domaine plan D une théorie (en partie valable sur une surface de Riemann) analogue à celle (de Hardy-F. Riesz, etc.) des fonctions f holomorphes dans le cercle unité et dont la moyenne de $|f|^p$ sur une circonférence concentrique variable est bornée. On étudie surtout la classe H_p des fonctions holomorphes f admettant dans D pour $|f|^p$ une majorante harmonique et on pose $\|f\|_p=[u(t)]^{1/p}$ où u est la valeur en un point fixé t de la plus petite majorante harmonique u de $|f|^p$ dans D. Soulignons que pour $p\geqq 1$ c'est une vraie norme, H_p est un espace de Banach et la convergence faible de f_n entraine la convergence uniforme locale dans D; H_2 peut être précisé et étudié comme espace de Hilbert. On étudie plus spécialement le cas de D limité par k courbes de Jordan analytiques disjointes. Chaque f de H_p admet alors presque partout à la frontière une limite non tangentielle définissant une fonction f^* pour laquelle la solution $\mathfrak{D}_f$ du problème de Dirichlet est précisément f. Ces fonctions-limite sur la frontière sont toutes les fonctions mesurables orthogonales (en dz sur la frontière) aux fonctions holomorphes sur $\bar{D}$ (c'est à dire dans un ouvert contenant D). La norme de f vaut la puissance $1/p$ de la valeur en t, $\mathfrak{D}(t)$, de la solution pour la donnée $|f^*|^p$. Enfin on étudie un peu différemment de Garabedian [mêmes Trans. **69**, 392–415 (1950); MR **12**, 492] les fonctions f nulles en t fixé, de norme $\|f\|_1\leqq 1$ (cette norme n'est pas celle de Garabedian). Si, pour ces fonctions f, $|f'(t)|$ admet en t une borne supérieure α, retenons surtout qu'il existe des f_0 de ce type telles que $f_0'(t)=\alpha$; elles sont holomorphes sur $\bar{D}$. On en étudie les zéros et leur connaissance détermine f_0. *M. Brelot* (Paris).

Maksimov, Yu. D. Extremal problems in certain classes of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1041–1044 (1955). (Russian)

Let $C_\epsilon(p;q)$ denote the class of functions $f(z)=z^q+\cdots$ $(q>0)$, regular in $|z|<1$, and for which there is a $\rho<1$ such that on each circle $z=re^{i\theta}$, $\rho<r<1$,

$$\int_{\theta_1}^{\theta_2}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta>-\epsilon,\ \theta_1<\theta_2;$$

$$\int_{-\pi}^{\pi}\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta=2p\pi.$$

Such functions are called locally ϵ-convex. Similarly he introduces the class $S_\epsilon(p;q)$ of locally ϵ-starlike functions, by replacing $1+zf''(z)/f'(z)$ with $zf'(z)/f(z)$ in the integrals.

The class $C_\epsilon(1;1)$ coincides with the close-to-convex univalent functions introduced by Kaplan [Michigan Math. J. **1**, 169–185 (1953); MR **14**, 966], and the classes $C_0(p;q)$, $S_0(p;q)$ coincide with those introduced by the reviewer [Trans. Amer. Math. Soc. **68**, 204–223 (1950); MR **11**, 508]. The author gives an integral representation for functions in these classes which contains those of Kaplan and the reviewer as special cases. Further he announces a very general theorem on extremal functions in these classes. As a special case these give sharp bounds for $|f'(z)|$ in the class $C_\epsilon(p;q)$ and for $|f(z)|$ in the class $S_\epsilon(p;q)$.

A. W. Goodman (Lexington, Ky.).

Pfluger, Albert. Extremallängen und Kapazität. Comment. Math. Helv. **29**, 120–131 (1955).

The following theorem is established. Let E denote an arbitrary set of the unit circumference $|z|=1$, let Γ_0 denote a Jordan curve in $|z|<1/3$ which is such that $z=0$ is contained in the bounded region with frontier Γ_0 and let J denote the totality of paths j (in $|z|<1$) which join Γ_0 to points of E. Let γ denote the Robin constant of E, and let λ denote the extremal length of the family J. Then there exist positive numbers k and K depending only on Γ_0, such that $2\gamma+k\leqq 2\pi\lambda\leqq 2\gamma+K$. In particular, $\lambda=+\infty$ if and only if the (outer) capacity of E vanishes. A theorem of Dufresnoy [Bull. Sci. Math. (2) **69**, 21–36, 117–121 (1945); MR **7**, 56, 379] concerning univalent conformal maps is established with the aid of the above theorem. The theorem is applied to show that quasiconformal maps of the interior of the unit circle onto itself induce mappings of the unit circumference onto itself with the property that a set of the unit circumference and its image both have outer capacity or outer capacity positive. The following result is also established: Let $w(z)$ be a non-constant meromorphic function in the interior of the unit circle whose Riemannian image has finite spherical area. If

$$S(t)=\int_{|w(z)|<t}|w'|^2(1+|w|^2)^{-2}dxdy=o(t^2\log t^{-1}),$$

then the set of points of $|z|=1$ at which $w(z)$ has the asymptotic value zero has zero outer capacity.

M. Heins (Providence, R. I.).

Fil'čakov, P. F. On the method of successive conformal mappings. Dokl. Akad. Nauk SSSR (N.S.) **101**, 25–28 (1955). (Russian)

Let D be the domain obtained by deleting from the lower half z-plane the points $(x-m)^2/a^2+y^2/b^2\leqq 1$. The function

$$w=S(m,a,b,z)\equiv\frac{a(z-m)-b((z-m)^2+b^2-a^2)^{1/2}}{a-b}$$

maps D onto the lower half w-plane with suitable normalization at $z=\infty$. The author asserts that by a suitable sequence of mappings $z_{n+1}=S(m_n,a_n,b_n,z_n)$ an arbitrary simply-connected domain can be mapped onto a half-plane. Although the reviewer cannot agree with this broad statement, the usefulness of such a sequence for domains D, which arise in certain applications, is amply demonstrated by the author who gives in detail a set of computations with $n=1,2,\cdots,12$ for a domain which occurs in a typical hydrodynamics problem. (The author directs the positive imaginary axis downward.) *A. W. Goodman.*

Warschawski, S. E. On mean convergence in conformal mapping. Arch. Math. 6, 102–114 (1955).

Suppose C and C_1 are two closed Jordan curves in the w-plane, both possessing continuously turning tangents and containing $w=0$ in their interiors. Denote by $\tau(s)$ the tangent angle of C expressed as a function of the arc length s. C is called of class $C'(\gamma, H)$ if for some γ, $0<\gamma\leqq 1$, and some $H>0$, $|\tau(s\pm t)-\tau(s)|\leqq Ht^{\gamma}$ for all points s of C. Suppose that C is of class $C'(\gamma, H)$ and that C_1 lies in the ϵ-neighborhood of C for some $\epsilon>0$. Denoting by $\tau^*(w)$ and $\tau_1^*(w)$ the tangent angles of C and C_1, respectively, as functions of the point w, suppose that there exists a constant $q>0$ such that for every point w_1 of C_1 there is a point w of C with $|w-w_1|\leqq\epsilon$ such that $|\tau^*(w)-\tau^*(w_1)|\leqq q\epsilon^{\gamma}$. Let $w=f(z)$ and $w=f_1(z)$ map $|z|<1$ conformally onto the interiors of C and C_1, respectively, so that $f(0)=f_1(0)=0$, $f'(0)>0$, $f_1'(0)>0$. If certain supplementary conditions are imposed on C and C_1, then for every $p>1$ there exists a constant M_p, depending only on certain quantities involved in the conditions imposed on C and C_1, such that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|f'(e^{i\theta})-f_1'(e^{i\theta})|^p d\theta\right\}^{1/p}\leqq M_p\epsilon^{\gamma}.$$

This result represents an extension of earlier results of the author's [Nachr. Ges. Wiss. Göttingen **1930**, 344–369; Proc. Amer. Math. Soc. **1**, 562–574 (1950); MR **12**, 170]. *W. Seidel* (Notre Dame, Ind.).

Tsuji, Masatsugu. The boundary distortion on conformal mapping. J. Math. Soc. Japan **6**, 235–261 (1954).

Let D be a domain in the $w(=\xi+i\eta)$-plane bounded by a Jordan curve through $w=0$, which touches the real axis in $w=0$ with inner normal at $w=0$ coincident with the positive η-axis. Let $w=w(z)$ map D conformally onto the upper half-plane of the $z(=x+iy)$-plane. The author's main theorem is the following: If in a neighborhood of $w=0$, C lies between two curves H and $\bar{H}$, each symmetric to $\xi=0$, and H: $\eta=h(\xi)$, $\bar{H}$: $\eta=-h(\xi)$ $(0\leqq\xi\leqq\delta)$ $h(0)=0$, where $\int_0^{\delta}h(t)t^{-2}dt<\infty$, then

$$\lim_{z\to 0}\frac{w(z)}{z}=\lim_{z\to 0}w'(z)=\gamma \quad (0<\gamma<\infty)$$

exists uniformly when $z\to 0$ in any Stolz domain with vertex at $z=0$. The author also obtains by the same methods another proof of a theorem of Warschawski [Math. Z. **35**, 321–456 (1932)]. Furthermore, if H: $\theta=\theta(r)$, $\bar{H}$: $\theta=-\theta(r)$, $0\leqq r\leqq\delta$, $\theta(0)=0$, such that

$$\int_0^{\delta}\theta(r)r^{-1}dr<\infty,$$

then $\lim w(z)/z=\gamma$, $0<\gamma<\infty$, uniformly when $z\to 0$ in any Stolz domain with vertex at $z=0$. When D lies in $\eta>0$, this was proved by Valiron [Bull. Sci. Math. (2) **56**, 208–211 (1932)]. When we only assume the tangency condition of the first sentence, he proves that

$$0<A|z|^{1+\epsilon}\leqq|w(z)|\leqq B|z|^{1-\epsilon},\quad 0<A|z|^{\epsilon}\leqq|w'(z)|\leqq B|z|^{-\epsilon},\quad z\in K_{\rho_0}(\varphi_0),$$

where $A>0$, $B>0$ are constants and $K_{\rho_0}(\varphi_0)$ denotes a sector which is bounded by a circle of radius ρ_0 about $z=0$ and two lines through 0 each making an angle φ_0 with the positive imaginary axis. He finally gives another proof and some applications of a theorem of Kellogg [Trans. Amer. Math. Soc. **13**, 109–132 (1912)]. *G. Springer.*

Brown, L. J. M. On conformal mappings of domains of infinite connectivity. Proc. Cambridge Philos. Soc. **51**, 56–64 (1955).

Poursuivant les recherches de H. Grötzsch et de H. Meschkowski, l'A. établit d'abord des conditions nécessaires et des conditions suffisantes nouvelles pour qu'une composante ponctuelle limite de la frontière soit parfaite au sens de Grötzsch [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **87**, 319–324 (1935)]. Puis il étudie la représentation conforme de certains domaines de connexion infinie sur les domaines limités par une suite de courbes F_n, resp. semblables à des courbes G_n, convexes, analytiques et de courbure bornée; et il étend à ces domaines les résultats obtenus par Meschkowski [Math. Ann. **124**, 178–181 (1952); MR **13**, 642] pour les domaines circulaires. *J. Lelong.*

Walsh, Joseph L. Sur la représentation conforme des aires multiplement connexes. C. R. Acad. Sci. Paris **239**, 1756–1758 (1954).

L'A. établit le résultat suivant, qui constitue un cas limite d'un précédent [mêmes C. R. **239**, 1572–1574 (1954); MR **16**, 581]. Toute aire D du plan des z, limitée par ν courbes de Jordan C_i, peut être représentée conformément sur une aire Δ du plan des Z, définie par:

$$\left|A\prod_1^{\mu}(Z-a_i)^{M_i}\prod_1^{\nu}(Z-b_j)^{-N_j}\right|<1$$

avec $\sum N_j=1$, les M_i étant des nombres positifs donnés satisfaisant à $\sum M_j=1$ et les images réciproques a_i des a_i étant fixées dans D. *J. Lelong* (Lille).

Walsh, Joseph L. Sur l'approximation par fonctions analytiques bornées. C. R. Acad. Sci. Paris **239**, 1339–1341 (1954).

The author proves a result which is related to some of his previous work [Proc. Nat. Acad. Sci. U. S. A. **24**, 477–486 (1938)]. He uses a new method. Let A be the annular region between the Jordan curves C_0 and C_1, where C_0 is interior to C_1; let $u(z)$ be a function harmonic in A and continuous on the closure of A, $=0$ or 1 on C_0 or C_1, respectively, and let C_σ $(0<\sigma<1)$ be the curve $u(z)=\sigma$ in A. Given a function $f(z)$ which is analytic in A and continuous on the closure of A, and satisfies a certain Lipschitz condition on C_σ, there exist functions $F_n(z)$ approximating to $f(z)$ on C_0 which are analytic in A and continuous on its closure, and such that $|f(x)-F_n(z)|$ satisfies some sharp inequality for z on C_0 and $F_n(z)$ for z in A. If $f(z)$ is analytic throughout the interior of C_σ, and C_0 is analytic, then the $F_n(z)$ can be chosen so that they are analytic throughout the interior of C_1. Again a converse theorem is stated. The results are proved by means of the conformal mapping of A on a circular ring and by the decomposition of a function into its components. The author states that the theorem admits of immediate applications to approximation by polynomials and by rational functions. *H. Kober.*

Kakehashi, Tetsujiro. The divergence of interpolations. I, II, III. Proc. Japan Acad. **30**, 741–745, 820–824, 963–964 (1954).

Let $f(z)$ be analytic in $|z|<R$, $R>1$ and possess a certain type of singularity on $|z|=R$. Set

$$W_{n+1}(z)=(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_{n+1}^{(n)}),$$

where the points $z_i^{(n)}$ lie in $|z|\leqq 1$. Assume that $W_n(z)/z^n$ converges in $|z|>1$ to a function which is analytic and non-vanishing there and that this convergence is uniform in

every closed set exterior to $|z| \leqq 1$. Then the sequence of polynomials $S_n(z; f)$ of degree n which interpolate to $f(z)$ in all the zeros of $W_{n+1}(z)$ converges to $f(z)$ for $|z| < R$ and diverges for $|z| > R$. The type of singularity considered is as follows: $f(z) = \varphi(z) + \sum_{k=1}^{N} \varphi_k(z) y_{m_k}(z; a_k)$, where φ and φ_k are analytic in $|z| \leqq R$, $|a_k| = R$ and

$$y_m(z; a) = \frac{\Gamma(1-m)}{2\pi i} \operatorname{Pf} \int_{C_R} \frac{\varphi(t)}{t-z} (t-a)^{m-1} dt, \quad |a| = R,$$

Pf = finite part. Generalizations are made to wider classes of regions. *P. Davis* (Washington, D. C.).

Kakehashi, Tetsujiro. On the convergence-region of interpolation polynomials. J. Math. Soc. Japan **7**, 32–58 (1955).

Given a sequence of polynomials $\varphi_n(z)$ of degree n which satisfies a certain growth condition. The author considers the problem of interpolation to analytic functions $f(z)$ at the zeros of $\varphi_n(z)$ by polynomials $P_n(z; f)$. Both the unit circle and more general domains are taken up. Convergence theorems and asymptotic expressions are obtained. Some of these can be regarded as generalizations of results of Walsh [Bull. Amer. Math. Soc. **42**, 715–719 (1936)]. Applications are made to the study of interpolation at the zeros of sets of polynomials $\varphi_n(z)$ which are orthogonal (with respect to a weight) on the unit circle and on $[-1, 1]$. For a weight on $[-1, 1]$ of the form $G(x)/(1-x^2)^{1/2}$, $G(x) > 0$ and analytic on $[-1, 1]$, $P_n(z; f)$ converges to f in the same ellipses associated with the Fourier expansion of f in φ_n's. Interpolation to functions possessing a finite number of poles is also discussed. *P. Davis* (Washington, D. C.).

Gaier, Dieter. Über Interpolation in regelmässig verteilten Punkten mit Nebenbedingungen. Math. Z. **61**, 119–133 (1954).

Let C be a closed Jordan curve in the complex plane and D its interior and let the function $\psi(w) = cw + c_0 + c_1 w^{-1} + \cdots$, $c \neq 0$, map $|w| > 1$ conformally on the exterior of C. Under this map extended to the boundaries the points

$$w_{nk} = \exp\{2\pi i k/n\}$$

on $|w| = 1$ correspond to points z_{nk} on C, for $n = 1, 2, \cdots$, $k = 1, \cdots, n$. Let f be a function defined on C and $\{a_{nk}\}$ a sequence of constants. The "nth interpolation polynomial" corresponding to f and $\{a_{nk}\}$ is the unique polynomial of degree $\leqq 2n-1$ which agrees at z_{nk} with f and has there derivative a_{nk}, $k = 1, \cdots, n$. Let C be an analytic curve, f integrable on C in the sense of Riemann and $a_{nk} = o(n/\log n)$ uniformly in k. If J_n is the nth interpolation polynomial corresponding, then for $z \in D$,

$$\lim_{n\to\infty} J_n(z) = (2\pi i)^{-1} \int_C f(\xi)(\xi - z)^{-1} d\xi$$

uniformly in each closed subset of D (Theorem 1). Next let f be analytic in D, continuous in $\bar{D}$, and let T_n be the Tchebysheff polynomial of degree $\leqq 2n-1$ corresponding to f on $\bar{D}$. Set $E_n = \max_{z \in \bar{D}} |f(z) - T_n(z)|$ and choose $a_{nk} = T_n'(z_{nk})$. Let F_n denote the nth corresponding interpolation polynomial. Then for each closed subregion B of D we can find a constant M with $|f(z) - F_n(z)| \leqq ME_n$ ($z \in B$, $n = 1, 2, \cdots$) and we can find a constant $N = N(C)$ with $|f(z) - F_n(z)| \leqq NE_n \log n$ ($z \in \bar{D}$, $n = 2, 3, \cdots$) (Theorem 2). Let now f be analytic in an annular region containing C bounded on the outside by C^1 and on the inside by C^2. Under a regularity condition on C, the author proves the following: Choose $a_{nk} = f'(z_{nk})$ and let H_n be the nth corresponding interpolation polynomial. Then

$$\lim_{n\to\infty} H_n(z) = (2\pi i)^{-1} \int_{C^1} f(\xi)(\xi - z)^{-1} d\xi$$

uniformly in every closed subregion of D (Theorem 3). Among the other results given in the paper are some counterexamples. In particular, it is shown that there exists a function analytic in $|z| < 1$ and continuous in $|z| \leqq 1$, $f(z)$, such that if L_n is the polynomial of degree $\leqq n-1$ with $L_n(z_{nk}) = f(z_{nk})$ with

$$z_{nk} = \exp\{2\pi i k/n\} \quad (k = 1, \cdots, n;\ n = 1, 2, \cdots),$$

then the sequence $L_n(-1)$ is not Borel-limitable. *J. Wermer* (Providence, R. I.).

Trošin, G. D. On the interpolation of functions analytic in a half-plane. Mat. Sb. N.S. **36(78)**, 39–56 (1955). (Russian)

The author considers the class of functions $f(z)$, regular in $x > 0$ and satisfying $|f(z)| < e^{ax}$, and asks the following two questions. (1) For which sequences $\{\lambda_n\}$ is there a function of the class such that $f(\lambda_n) = a_n$ whenever

$$|a_n| < \exp\{a\Re(\lambda_n)\}?$$

(2) Given $\{\lambda_n\}$, for which a_n is there a function such that $f(\lambda_n) = a_n$? Here $|\lambda_n|$ increases, $\lambda_n \to \infty$, and λ_n are in a closed angle interior to the right-hand half-plane. The answer to (1) is that necessary and sufficient conditions are $\sum 1/|\lambda_n| < \infty$ and $\limsup |\lambda_n|^{-1} \log |1/\Phi'(\lambda_n)| < \infty$, where $\Phi(z) = \prod (1 - z/\lambda_n)/(1 + z/\bar{\lambda}_n)$. The answer to (2) is given in terms of a certain sequence of exponential polynomials, whose convergence furnishes a necessary and sufficient condition for the existence of the required function. *R. P. Boas, Jr.* (Evanston, Ill.).

Radojčić, M. Sur les séries de fonctions algébriques et les produits infinis analogues, définissant des fonctions analytiques multiformes dans leurs domaines d'existence quelconques. Acad. Serbe Sci. Publ. Inst. Math. **7**, 95–118 (1954).

H. Florack established [Schr. Math. Inst. Münster no. 1 (1948); MR **12**, 251] a generalization of the theorems of Weierstrass and Mittag-Leffler to non-compact Riemann surfaces and at the same time showed the existence of a non-continuable analytic function whose domain of existence is a given non-compact Riemann surface. This work is based on the results of Behnke and Stein [Math. Ann. **120**, 430–461 (1949); MR **10**, 696]. The present paper treats the same problems by methods developed earlier by the author [C. R. Acad. Sci. Paris **185**, 1007–1009 (1927)]. *M. Heins* (Providence, R. I.).

Andreian, Cabiria. Relations de structure dans la famille des transformations intérieures. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 431–441 (1953). (Romanian. Russian and French summaries)

Extension d'un théorème de Stoïlov [Leçons sur les principes topologiques . . . , Gauthier-Villars, Paris, 1937, p. 121] au cas d'une projection intérieure d'une surface de Riemann V sur une autre surface de Riemann V_0. Soit $\mathfrak{I}$ l'ensemble des transformations intérieures $V - V_0$, $\mathfrak{F} \subset \mathfrak{I}$ l'ensemble des projections riemanniennes F appliquant V sur V_0, $\mathfrak{T}$ l'ensemble des représentations topologiques $V \rightleftarrows V$ et $\mathfrak{C} \subset \mathfrak{T}$ l'ensemble des transformations conformes biunivoques $V \rightleftarrows V$. Soit $\mathfrak{B}$ l'ensemble des surfaces de Riemann

V_τ homéomorphes à V et T_τ une transformation topologique qui applique V_τ sur V. On définit d'une manière analogue $\mathfrak{J}_\tau$, $\mathfrak{F}_\tau$, $\mathfrak{T}_\tau$, et $\mathfrak{C}_\tau$.

Posant alors $\mathfrak{J}_V{}^V=\mathfrak{F}\mathfrak{T}$, $\mathfrak{J}_{V_\tau}{}^{V_\tau}=\mathfrak{F}_\tau\mathfrak{T}_\tau$, $\mathfrak{J}_V{}^{V_\tau}=\mathfrak{J}_{V_\tau}{}^{V_\tau}\mathfrak{T}_\tau^{-1}$, le théorème de Stoïlov s'écrit

$$\mathfrak{J}=\bigcup_{V_\tau\in\mathfrak{B}}\mathfrak{J}_V{}^{V_\tau},\quad \mathfrak{J}_V{}^{V_\tau}\cap\mathfrak{J}_V{}^{V_k}=0\quad(k\neq\tau).$$

Considérant alors les classes $\mathfrak{J}_V{}^V/\mathfrak{T}$ et d'autres du même type, l'auteur établit les correspondences biunivoques

$$\mathfrak{J}_V{}^V/\mathfrak{T}\rightleftarrows\mathfrak{F}/\mathfrak{C}\quad\text{et}\quad\mathfrak{J}_V{}^V/\mathfrak{F}\rightleftarrows\mathfrak{T}/\mathfrak{C}$$

qu'il généralise ensuite par l'application du théorème de Stoïlov: on obtient en particulier

$$\mathfrak{J}/\mathfrak{T}\rightleftarrows\bigcup_{V_\tau\in\mathfrak{B}}\Phi_\tau/\Gamma_\tau\quad\text{où}\quad\Phi_\tau=\mathfrak{F}_\tau T_\tau^{-1}\quad\text{et}\quad\Gamma_\tau=T_\tau\mathfrak{C}_\tau T_\tau^{-1}.$$

L. Fourès (Marseille).

Hopf, Heinz. Schlichte Abbildungen und lokale Modifikationen 4-dimensionaler komplexer Mannigfaltigkeiten. Comment. Math. Helv. **29**, 132–156 (1955).

Une représentation $y=f(x)$ d'un domaine de $C^2(x_1, x_2)$ ou d'une variété à structure analytique complexe X dans $C^2(y_1, y_2)$ sera dite univalente (schlicht) sauf pour $y=a$ si f transforme biunivoquement $X-A$; $A=f^{-1}(a)$ est nécessairement un point ou une variété de dimension complexe un en tous ses points; la transformation f est dite complète si a est point intérieur de $f(x)$. Les modifications locales en a d'une variété Y en une variété X, au sens de Behnke et Stein [Math. Ann. **124**, 1–16 (1951); MR **13**, 644] s'identifient aux représentations f^{-1}, où f est univalente sauf en a. On sait qu'une modification locale où $A=f^{-1}(a)$ est compact peut être obtenue par une suite finie de modifications σ (éclatements de Hopf) consistant à remplacer un point par un espace projectif ou sphère [H. Hopf, Rend. Mat. e Appl. (5) **10**, 169–182 (1951); MR **13**, 861; voir aussi, O. Zariski, Ann. of Math. (2) **45**, 472–542 (1944), pour le cas algébrique; MR **6**, 102]. On donne ici des procédés variés de modification locale à l'aide de modifications σ et une démonstration de l'énoncé précédent.

Les germes irréductibles de A n'ont que des points ordinaires; par un point il en passe deux au plus, qui ne peuvent être tangents. Si k est un compact contenu dans $A\subset X$, il existe une variété $\bar{Y}$, biunivoque au voisinage de K, avec $f=\Omega\circ F$, où Ω est la projection $\bar{Y}\to Y$ et où $K\to f(K)\subset\bar{\Sigma}$ est biunivoque, $\bar{\Sigma}$ étant l'ensemble des sphères issues des éclatements σ. Etude du cas où f est complète au sens indiqué plus haut: sauf si f est biunivoque, f ne peut être complète quand X est C^2 ou un domaine de C^2.

P. Lelong (Paris).

Stoll, Wilhelm. Über meromorphe Modifikationen. II. Allgemeine Eigenschaften meromorpher Modifikationen. Math. Z. **61**, 467–488 (1955).

Les notations sont celles d'un travail précédent [Math. Z. **61**, 206–234 (1954); MR **16**, 689] auquel nous renvoyons. Une représentation f d'une variété G^n dans une variété H^m (n, m sont des dimensions complexes) est dite méromorphe si elle est analytique complexe sur un ouvert de G^n et si, essentiellement, la trace de f sur les courbes L^1 (variété à une dimension complexe de G^n) est méromorphe. Une modification $\mathfrak{M}=\mathfrak{M}(G, A, M, \tau;\ H, B, N, \upsilon)$ est méromorphe si τ est méromorphe. Pour $n=2$,

$$\mathfrak{M}^{-1}=\underset{2}{\mathfrak{M}}(H, B, N, \upsilon;\ G, A, M, \tau)$$

est aussi méromorphe. Une modification est dite ouverte si pour tout ouvert $U\supset M$, $\tau[(U\cap A)\cup N]$ est ouvert; rappelons que G et H sont deux variétés de dimension complexe n, les ensembles modifiés M (N) sont fermés; $M\subset G$ ($N\subset H$) sont contenus dans des variétés analytiques de dimension $n-1$ au plus; $\tau=\upsilon^{-1}$ est une transformation pseudo-conforme de $A=G-M$ en $B=H-N$. $\underset{n}{\mathfrak{M}}$ méromorphe et ouverte permet de prolonger de G à H une fonction méromorphe au voisinage de $M\subset G$. On donne des énoncés assez nombreux permettant de prolonger τ (ou υ) à des partiés de A (ou B), du type suivant: $\Sigma(P_0, L_i)$ désigne l'éclatement en P_0 de τ le long de la courbe $L_i{}^1$, c'est-à-dire $Q\in\Sigma(P_0, L_i)$ si $Q=\lim\tau(P_\nu)$, $P_\nu\in L_i\cap A$, $P_\nu\to P_0$; si $\underset{n}{\mathfrak{M}}$ est ouverte, υ régulière en $Q_0\in N_1$, les $L_i{}^1$ étant n courbes analytiques régulières en P_0 avec $M\cap L_i=P_0$, à tangentes linéairement indépendantes en P_0, alors si $Q_0\in\bigcap_{i=1}^n\Sigma(P_0, L_1{}^i)$ $\underset{n}{\mathfrak{M}}$ ne modifie pas G en P_0; on étend ainsi à $n>2$ quelques uns des résultats du récent mémoire de H. Hopf analysé ci-dessus.

P. Lelong (Paris).

Schwartz, Marie-Hélène. Formules apparentées à celles de Nevanlinna-Ahlfors pour certaines applications d'une variété à n dimensions dans une autre. Bull. Soc. Math. France **82**, 317–360 (1954).

This paper generalizes the Ahlfors theory of covering surfaces [Acta Math. **65**, 157–194 (1935)] to manifolds of n-dimension. Let f be an interior mapping of an n-dimensional C_2 manifold V onto a compact C_2 manifold W. Furnish W with a Riemannian metric. Then the mapping f is called parabolic with respect to an exhaustion $\{V_i\}$ of V if $\lim\mathfrak{B}(\dot{V}_i)/\mathfrak{B}(V_i)=0$, where $\mathfrak{B}$ and $\dot{\mathfrak{B}}$ are the n- and $(n-1)$-dimensional volumes in V inherited by f from the Riemannian metric on W, and where $\dot{V}_i$ denotes the boundary of V_i.

The author first establishes the existence of a function $N(t)$ which measures the number of times "most" points of W are taken by f in V_t. She then establishes a relation between $N(t)$, the characteristic of the mapping of V_t into W, the Euler characteristic of W and a "curvature" integral over the boundary of V_t. If $n=2$ this becomes the Ahlfors theory. Further discussion of parabolic mappings is carried out in the paper.

H. L. Royden (Stanford, Calif.).

Popovici, Constantin. On the continuity of functions of several complex variables. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **5**, 477–480 (1953). (Romanian)

Pompeiu has shown [Dissertation, Paris, 1905] that there exist uniform analytic functions of a complex variable, continuous on a set of singularities of positive plane measure. The author generalizes this result to the case of several complex variables. Let $F(z)$ be a uniform, analytic function, continuous on a certain set E of singular points, $mE>0$. Let $\varphi_1(u_1, v_1; Z)$ be any function, continuous in the real variables u_1, v_1 and analytic in Z. Set $Z=F(z)$ and consider

$$\phi(z, z_1)=(2\pi i)^{-1}\int\int\varphi_1(u_1, v_1; F(z))(w_1-z_1)^{-1}d\omega,$$

where the integral is extended over the set E_1 in the w_1-plane, identical to the set E, and $d\omega$ is the element of area. By Pompeiu's theorem it follows, that ϕ is analytic and continuous on the set E_1 ($mE_1>0$) of its singular points. Using a theorem of Montel [Dissertation, Paris, 1907] it follows

that ϕ is analytic in z and z_1 jointly, whence, by iteration, the result: There exist uniform, analytic functions of n complex variables, continuous on a set of singular points of positive $2n$-dimensional measure. The result extends naturally to sets of singular points of $2q$-dimensional positive measure ($q \leqq n$). A similar result holds also for sets E of singularities of odd-dimensional positive measure, the corresponding functions being only bounded on E.

E. Grosswald (Philadelphia, Pa.).

Theory of Series

Mazur, S., and Orlicz, W. On linear methods of summability. Studia Math. **14**, (1954), 129–160 (1955).

In 1932 the authors made substantial applications of functional analysis to problems involving matrix methods for evaluation of sequences. At that time only a short summary [C. R. Acad. Sci. Paris **196**, 32–34 (1933)] of their principal results was published. During the intervening years a full account of this work remained unpublished while numerous other authors published in the field and in some cases rediscovered results of Mazur and Orlicz. The paper under review is a full and integrated account of the subject which includes work done by Mazur and Orlicz in 1932 and 1935 and by others since 1932. The definitions, theorems, and examples are so numerous and varied that we can only indicate the nature of the problems treated. Let A denote a matrix a_{nk} by means of which a given real sequence $x_0, x_1, x_2, \cdots$ is evaluable to $A(x)$ if the series in $\sigma_n = \sum_{k=0}^{\infty} a_{nk}x_k$ all converge and $\sigma_n \to A(x)$ as $n \to \infty$. Many of the results apply to matrices which are conservative (convergence-preserving) as well as to the smaller class of matrices which are regular (permanent). Some theorems involve the structure of the class A^* of sequences evaluable A, and some of these involve the question whether A^* contains unbounded sequences and the question whether the sequences in A^* must satisfy a condition on rate of growth. Some theorems on consistency give conditions on matrices A and B and a class X of sequences x under which $A(x) = B(x)$ whenever $x \in XA^*B^*$, and some of these involve the condition $B^* \supset A^*$. Sometimes X is the class of all sequences and sometimes it is the class of all bounded sequences. A variant and a generalization of the following fundamental theorem are given. If A and B are regular, and if each bounded sequence evaluable A is also evaluable B, then A and B are consistent for bounded sequences. An explicit and simple example exhibits two regular inconsistent matrices A and B for which $A^* = B^*$. Numerous other results include the following. If A is conservative and is such that the product sequence $x_n y_n$ is evaluable A whenever x_n and y_n are both evaluable A, then there is an increasing sequence $k_1, k_2, \cdots$ such that A^* is the class of sequences for which $\lim x_{k_n}$ exists.

R. P. Agnew (Ithaca, N. Y.).

Włodarski, L. Sur les méthodes continues de limitation. I. Application de l'espace B_0 de Mazur et Orlicz à l'étude des méthodes continues. Studia Math. **14** (1954), 161–187 (1955).

Włodarski, L. Sur les méthodes continues de limitation. II. Limitation des suites bornées. Studia Math. **14** (1954), 188–199 (1955).

A sequence $\lambda_0(t), \lambda_1(t), \cdots$ of functions defined over a set T having a limit point t_0 determines a sequence-to-function transformation and a method Λ by which a given sequence $x = x_0, x_1, x_2, \cdots$ is evaluable to $\Lambda(x)$ if the series in

$$\Lambda(t, x) = \sum_{k=0}^{\infty} \lambda_k(t)x_k$$

converge for each t in T and $\Lambda(t, x) \to \Lambda(x)$ as $t \to t_0$. While many theorems involving matrix transformations have well known and substantially identical analogs in the class Λ, it seems that some of the theorems of the paper of Mazur and Orlicz reviewed above are exceptions. In any case, analogs of several of the theorems of Mazur and Orlicz are proved for methods that are of type Λ and are "continuous" according to the following definition. A method of type Λ is "continuous" if (i) T is a real interval $0 \leqq t < t_0$; (ii) the functions $\lambda_k(t)$ are all continuous over T; and (iii) there exists an increasing sequence $t_1, t_2, \cdots$ such that $t_1 = 0$, $t_n \to t_0$, and the series $\sum_{k=0}^{\infty} \lambda_k(t)x_k$ is uniformly convergent over the interval $t_k \leqq t \leqq t_{k+1}$ whenever k and the sequence $x_0, x_1, \cdots$ are such that the series converges when $t = t_k$ and $t = t_{k+1}$. The class of "continuous" methods includes the familiar methods of Abel and Borel, and others for which $\lambda_k(t)$ is of the form $g(t)c_k t^k$.

R. P. Agnew (Ithaca, N. Y.).

Sikorski, R. A remark on the Mazur-Orlicz theory of summability. Bull. Acad. Polon. Sci. Cl. III. **3**, 11–15 (1955).

For each $q = 0, 1, 2, \cdots$, let T_q be the transformation which transforms a sequence $x = (x_0, x_1, \cdots)$ into the sequence

$$T_q(x) = (x_0, x_1, \cdots, x_q, 0, 0, \cdots).$$

For each t in a set T having a limit point t_0 not belonging to T, let

$$f(t, T_q(x)) = \sum_{k=0}^{q} a_k(t)x_k,$$

where $a_0(t), a_1(t), \cdots$ is a given sequence of functions. The sequence x is then evaluable to $f(x)$ if

$$f(x) = \lim_{t \to t_0} \lim_{q \to \infty} f(t, T_q(x)).$$

Instead of defining the transformations T and the functionals f as above, the author assumes that T and f are abstractly defined and satisfy conditions too lengthy for presentation here. He is then able to obtain some of the results of the three preceding papers. The main result is a variant of the fundamental theorem of Mazur and Orlicz on consistency in the class of bounded sequences. As the author points out, his proof is simpler than that of Mazur and Orlicz but his conditions on T and f are so restrictive that his result does not imply that of Mazur and Orlicz.

R. P. Agnew (Ithaca, N. Y.).

Čelidze, V. G. On the summation of double integrals. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 131–143 (1954). (Russian)

When $S(t, u)$ and $F(x, y)$ are defined by

$$S(t, u) = \int_0^t \int_0^u f(v, w)dvdw,$$

$$F(x, y) = \frac{1}{xy} \int_0^x \int_0^y S(t, u)dtdu,$$

it is shown that

$$\lim_{x, y \to \infty}{}^{*} F(x, y) = \lim_{t, u \to \infty} S(t, u)$$

whenever the right side exists, $S(t, u)$ satisfies stated conditions, and $\lim^*$ indicates that x and y become infinite subject to stated restrictions. There is an analogous result in which the Cesàro transform $F(x, y)$ is replaced by an Abel transform. *R. P. Agnew* (Ithaca, N. Y.).

Anastassiadis, Jean. Sur la convergence des séries de fonctions absolument continues. Bull. Sci. Math. (2) **78**, 234–240 (1954).

Principal result: if $f_1, f_2, \cdots$ are absolutely continuous real-valued functions on the closed real interval $[a, b]$, and if the series

$$\sum_{n=1}^{\infty} f_n(a), \quad \sum_{n=1}^{\infty} V_n(b)$$

are each convergent (where $V_n(x)$ is the total variation of f_n on $[a, x]$, then

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[a, b]$ to an absolutely continuous limit function. *T. A. Botts* (Charlottesville, Va.).

Slepenčuk, K. M. On a property of infinite products. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 151–153 (1955). (Russian)

This paper characterizes the real convergent infinite products $\prod_{k=1}^{\infty}(1+a_k)$ for which $\prod_{k=1}^{\infty}(1+a_kx)$ converges when x is real and $x \neq 0, 1$. The following modification of the work of the author removes the restriction that x is real.

A clue to the solution lies in the elementary fact that if a_k is real, $a_k \to 0$, and x is real or complex, then there is an index N and a sequence ϵ_k such that $\epsilon_k \to 0$ and

$$(*) \quad \prod_{k=m}^{n}(1+a_kx) = \exp\left[x\sum_{k=m}^{n}a_k\right]\exp\left[x^2\sum_{k=m}^{n}(-\tfrac{1}{2}+\epsilon_k)a_k^2\right]$$

whenever $N \leqq m \leqq n$. If $\prod(1+a_k)$ converges, then the two series $\sum a_k$ and $\sum a_k^2$ both converge, or both diverge to $+\infty$. In case $\prod(1+a_k)$ converges and the two series $\sum a_k$ and $\sum a_k^2$ both converge, use of (*) shows that $\prod(1+a_kx)$ converges for each complex x. In case $\prod(1+a_k)$ converges and the two series $\sum a_k$ and $\sum a_k^2$ both diverge to $+\infty$, use of (*) shows that the limit

$$(**) \quad \lim_{n\to\infty}\left[x\sum_{k=m}^{n}a_k + x^2\sum_{k=m}^{n}(-\tfrac{1}{2}+\epsilon_k)a_k^2\right]$$

exists when $x=1$; it follows from this that the limit (**) cannot exist when x is a complex number different from 0 and 1 hence from (*) that $\prod(1+a_kx)$ diverges when x is a complex number different from 0 and 1. *R. P. Agnew.*

Franklin, J., and Friedman, B. A convergent asymptotic representation for integrals. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-9**, i+17 pp. (1954).

By means of integration by parts the authors find under general conditions for $I_h(f) = \int_0^{\infty} e^{-px}x^{c-1+h}f_h(x)dx$, where $\mathrm{Re}\, c > 0$ and $h = 0, 1, \cdots$, a recurrence relation of the form

$$I_h(f_h) = f_h((c+h)p^{-1})\Gamma((c+h)p^{-1})p^{-c-h} + p^{-1}I_{h+1}(f_{h+1}),$$

where

$$f_{h+1}(x) = \frac{d}{dx}\,\frac{f_h(x) - f_h(c+h)p^{-1})}{x-(c+h)p^{-1}}.$$

This recurrence relation gives for $I_h(f)$, where $f(x) = f_0(x)$, a formal series

$$\sum \Gamma(c+h)p^{-2h-c}f_h((c+h)p^{-1}).$$

From the treated numerical examples it follows that the first term of this series alone yields, even for small $|p|$, an excellent approximation to the integral $I_0(f)$ in the three cases $c=1$, $f(x)=(1+x)^{-1}$; $c=1$, $f(x)=(1+x)^{-1/2}$ and $c=1/2$, $p=-iy$; $f(x)=(2+x)^{-1/2}$. In this last case, which treats the Hankel function $I_0(f) = e^{-iy}H_0^{(1)}(y)$, the factors $\exp(-i \arctan(4y)^{-1})$ on page 5 must be replaced by $\exp(-\frac{1}{2}i \arctan(4y)^{-1})$. If $f(x)$ is analytic for $\mathrm{Re}\, x > 0$ and if $|f^{(h)}(x)| \leqq Me^{\mu|x|}$ $(h = 0, 1, \cdots)$, where M and μ may depend on h, but not on x, then the formal series obtained for $I_0(f)$, represents, for large positive p, an asymptotic expansion for $I_0(f)$; in the special case that c is positive, the assumption that $f(x)$ is analytic for $\mathrm{Re}\, x > 0$, may be replaced by the weaker condition that $f(x)$ is infinitely differentiable for $x > 0$. If $f(x) = \int_0^{\infty} e^{-xt}dY(t)$ for $\mathrm{Re}\, x > 0$, where $Y(t)$ is of bounded variation in each finite interval $(0, T)$ and where $Y(t)$ is bounded for $t \geqq 0$, then the series, obtained for $I_0(f)$, converges to $I_0(f)$.

J. G. van der Corput (Berkeley, Calif.).

Mandelbrojt, S. Influence des propriétés arithmétiques des exposants dans une série de Dirichlet. Ann. Sci. Ecole Norm. Sup. (3) **71**, 301–320 (1954).

Die Dirichletsche Reihe $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ $(s = \sigma + it)$ mit $0 < \lambda_n \nearrow \infty$ und $\inf(\lambda_{n+1} - \lambda_n) = \alpha > 0$ habe die Konvergenzabszisse c $(-\infty < c < +\infty)$; die in $\sigma > c$ dargestellte Funktion sei $f(s)$. Für ein $a < c$ existiere ein grösstes Gebiet $R_a \supset \{\sigma > c\}$ in $\sigma > a$ mit der Eigenschaft, dass $f(s)$ eindeutig in R_a fortsetzbar ist. Es werden vier Sätze bewiesen, die das analytische Verhalten von $f(s)$ in R_a mit der Verteilung der λ_n in Zusammenhang bringen. Die Voranzeige dieser Ergebnisse [C. R. Acad. Sci. Paris **236**, 1464–1466 (1953)] ist schon früher besprochen worden [MR **14**, 1075].

D. Gaier (Stuttgart).

Fourier Series and Generalizations, Integral Transforms

Stečkin, S. B. Some remarks on trigonometric polynomials. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 159–166 (1955). (Russian)

Let $K_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos kt$. It is known that there are constants C_1, C_2 such that

$$\left|\sum_{k=0}^{n}\frac{a_k}{n-k+1}\right| \leqq C_1\int_0^{\pi}|K_n(t)|\,dt,$$

$$\sum_{k=0}^{n}\frac{|a_k|}{n-k+1} \leqq C_2\int_0^{\pi}|K_n(t)|\,dt$$

[Hille and Tamarkin, Trans. Amer. Math. Soc. **34**, 757–783 (1932); Sidon, J. London Math. Soc. **13**, 181–183 (1938)]. The author shows that $C_1 \leqq \int_0^{\pi} x^{-1}\sin x\,dx = 1.85\cdots$, while $C_2 \leqq 2$; and that under the additional requirement that $a_k \geqq 0$, the exact value of C_2 is $\pi/2$. He comments on the calculation of $\int_0^{\pi}|K_n(t)|\,dt$ by complex-variable methods and illustrates his remarks by obtaining a new formula for the Lebesgue constants. *R. P. Boas, Jr.* (Evanston, Ill.).

Raisbeck, Gordon. The order of magnitude of the Fourier coefficients in functions having isolated singularities. Amer. Math. Monthly **62**, 149–154 (1955).

As a sort of converse of Salem's theorem [C. R. Acad. Sci. Paris **186**, 1804–1806 (1928); Essais sur les séries trigonometriques, Hermann, Paris, 1940; MR **2**, 93, 419], the author proves that if $f(x)$ is a positive, non-constant, non-

increasing and convex function in the interval $(0, \pi)$ and $xf(x)$ is there integrable, then

$$n\int_0^{\pi/n} xf(x)dx > \int_0^{\pi} f(x)\sin nx dx > cn\int_0^{\pi/2n} xf(x)dx,$$

where c is a constant independent of n. From this he derives a similar formula for cosine integrals. He gives further some applications. *S. Izumi* (Tokyo).

Syneček, Vladimír. A contribution to the inequality and equality relations in structure analysis. Czechoslovak J. Phys. 4, 472–477 (1954). (Russian summary)

The author gives a concise derivation and discussion of some inequalities for the coefficients in the (three-dimensional) Fourier series of a nonnegative function.

R. P. Boas, Jr. (Evanston, Ill.).

Verblunsky, S. On the roots of a transcendental equation, occurring in the theory of trigonometric series. Math. Z. 61, 324–335 (1954).

This paper is a continuation of previous work by the author [Rend. Circ. Mat. Palermo (2) 3, 89–105 (1954); MR 15, 953]. Let A and B be relatively prime polynomials with real coefficients and let the equation

(i) $$A(z)\sin \pi z - B(z)\cos \pi z = 0$$

have all its roots real and simple. If $f(x)$ is integrable in $(\alpha, \alpha+2\pi)$, set

$$\alpha_k - i\beta_k = \gamma_k \int_\alpha^{\alpha+2\pi} f(t)e^{-i\mu_k t}dt,$$

where γ_k is the residue of

$$Q = \frac{1}{2}\frac{A\cos \pi z + B\sin \pi z}{A\sin \pi z - B\cos \pi z}$$

at μ_k. Further let $0 < \delta \neq |\mu_k - [\mu_k]|$ for all μ_k and set

$$s_n(x) = \sum \alpha_k \cos \mu_k x + \beta_k \sin \mu_k x,$$

the sum being extended over all $|\mu_k| < n+\delta$. Then $s_n(x)$ is equiconvergent with the partial sums of the Fourier series of a function obtained from $f(x)$ by a simple procedure. The simplest case is when degree of $A <$ degree of B. Here $s_n(x)$ and the nth partial sum of the Fourier series of $f(x)$ are uniformly equiconvergent in the closed interval $(\alpha, \alpha+2\pi)$.

The roots of (i) are discussed and necessary and sufficient conditions are given in order that they be real and simple under the assumption that the degrees of A and B are ≤ 1, or that one of the polynomials is even and quadratic and the other odd and linear. *A. P. Calderón.*

Salem, R. On strong summability of Fourier series. Amer. J. Math. 77, 393–403 (1955).

Let (n_k) be an increasing sequence of integers such that (i) $n_k = O(k^4)$ for some A, (ii) $(n_{k-1} - n_k)/n_k > \epsilon/k$ for some $\epsilon > 0$, and (iii) there is a sequence (k_ν) with an Hadamard gap such that $n_{k_{\nu+1}}/n_{k_\nu}$ is bounded. The author proves that if

$$\int_0^\eta |\varphi_x(t)|^s dt = o(\eta), \quad \varphi_x(t) = f(x+t)+f(x-t)-2f(x),$$

then the sequence $(s_{n_k}(x))$, $s_n(x)$ being the nth partial sum of Fourier series of $f(x)$, is strongly summable to $f(x)$, that is,

$$k^{-1}\sum_{j=1}^{k} |s_{n_j}(x) - f(x)|^s \to 0 \quad (k \to \infty)$$

for any $r > 0$. There are no theorems of this type in the literature, except in the case $n_k = k, k^2, k^3, \cdots$. He proves further that there is a continuous function $f(x)$ such that

$$n^{-1}\sum_{k=1}^{n} \max_x |s_k(x)|$$

is unbounded. *S. Izumi* (Tokyo).

Džvaršeišvili, A. G. On the summation of double trigonometric series by a method of Riemann. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 20, 157–166 (1954). (Russian)

Terminology and results of V. G. Čelidze [same Trudy 17, 61–94 (1949); MR 12, 820] and K. K. Gahariya [Mat. Sb. N.S. 28(70), 337–350 (1951); MR 12, 822] are used. It is shown that if a double trigonometric series converges almost everywhere to $f(x, y)$, and if the terms and partial sums of the series satisfy stated conditions, then the series is restrictedly evaluable almost everywhere to $f(x, y)$ by a Riemann method of order 2. Closely related results are obtained and used to obtain the following uniqueness theorem. If a double trigonometric series converges almost everywhere to 0, and if the terms and partial sums of the series satisfy stated conditions, then the terms of the series are all 0. *R. P. Agnew* (Ithaca, N. Y.).

Shapiro, Victor L. Localization of conjugate multiple trigonometric series. Ann. of Math. (2) 61, 368–380 (1955).

Let $T = \sum_m a_m e^{i(m\cdot x)}$, where

$$x = (x_1, x_2, \cdots, x_n), \quad (x\cdot m) = x_1m_1 + x_2m_2 + \cdots + x_nm_n$$

and m runs over all points in E_n with integral coordinates, be a trigonometrical series in n variables. Let $K(x)$ be a spherical harmonic of degree g. Then

$$T^* = \sum_m K(m)|m|^{-\nu}a_m e^{i(m\cdot x)}, \quad |m|^2 = (m\cdot m),$$

is called the series conjugate to T with respect to K. If $T_1 = \sum a_m e^{i(m\cdot x)}$, $T_2 = \sum b_m e^{i(m\cdot x)}$, and if $c_m = \sum_k a_{m-k}b_k$ converges absolutely for all m, then $\sum c_m e^{i(m\cdot x)}$ is called the formal product of T_1 and T_2.

The author uses the theory of formal multiplication in order to obtain localization properties for T^*. The main result is the following. If $|a_m| = o(|m|^\gamma)$, $\gamma \geqq -n+1$, and the series $\sum a_m |m|^{-2\omega} e^{i(m\cdot x)}$, ω being the integral part of $\frac{1}{2}(n+\gamma)+1$, represents a function in $C^{(2\omega+n+2)}$ on a domain D, then T^* is uniformly spherically summable $(C, \gamma+n-1)$ in every closed set C contained in D, that is

$$\sum_{|m|\leq R} K(m)|m|^{-\nu}(1-|m|^2R^{-2})^{\gamma+n-1}e^{i(m\cdot x)}$$

converges uniformly in C as $R \to \infty$.

An interesting auxiliary result proved in the paper is this: if $T_1 = \sum a_m e^{i(m\cdot x)}$, $|a_m| = o(|m|^\gamma)$, $\gamma \geqq -n+1$, and $T_2 = \sum b_m e^{i(m\cdot x)}$ is the Fourier series of a function in C^∞ vanishing with all its derivatives on a closed set E, then the series T_2^* conjugate to T_1T_2 is uniformly spherically summable $(C, \gamma+n-1)$ on E. *A. P. Calderón.*

Calderón, A. P., and Zygmund, A. On a problem of Mihlin. Trans. Amer. Math. Soc. 78, 209–224 (1955).

Let $x, y, \cdots$ denote points or vectors in k-dimensional Euclidean space E^k; let $|x|$ denote the length of the vector x, and write x' for $x/|x|$ $(x \neq 0)$. Let $K(x, y)$ be a function of the form $\Omega(x, z')/|z|^k$, where $z = x - y$. Suppose that

(1) $$\int_\Sigma \Omega(x, z')dz' = 0$$

for all x, and that

$$\int_\Sigma |\Omega(x, x')|^2 dx' \leqq A, \tag{2}$$

where A is independent of x; Σ denotes the $(k-1)$-sphere $|x|=1$. Let $f \in L^2(E^k)$. The authors show that, under these conditions, the integral

$$f_\epsilon(x) = \int_{|x| \geqq \epsilon} K(x, y) f(y) dy$$

is absolutely convergent for all x and for all $\epsilon > 0$, and that there is an L^2 function $\tilde{f}(x)$ such that $\tilde{f}_\epsilon(x)$ converges to $\tilde{f}(x)$ in the L^2 sense when $\epsilon \to +0$. Furthermore,

$$\|\tilde{f}_\epsilon\| \leqq A\|f\|, \quad \|\tilde{f}\| \leqq A\|f\|,$$

where A is independent of the particular function f.

The main tools of the proof are expansions in ultraspherical harmonics and the k-dimensional Fourier transform. These enable the authors to reduce the problem to the proof of the inequality

$$\left| \int_\lambda^\infty \frac{J_{n+\lambda}(\rho)}{\rho^{1+\lambda}} d\rho \right| \leqq \frac{A}{n^{1+\lambda}},$$

where $\lambda = \frac{1}{2}k - 1$, $k > 0$, $n \geqq 1$, and A is independent of k and n.

A refinement of the method of proof enables the authors to show that (2) can be replaced by

$$\int_\Sigma |\Omega(x, x')|^p dx' \leqq A,$$

for $p > 2 - (2/k)$. *F. Smithies* (Cambridge, England).

*Besicovitch, A. S. **Almost periodic functions.** Dover Publications, Inc., New York, 1955. xiii+180 pp. Paperbound \$1.75; clothbound \$3.50.

Reprint by photo-offset of the 1st edition [Cambridge, 1932].

Eberlein, W. F. A note on Fourier-Stieltjes transforms. Proc. Amer. Math. Soc. 6, 310–312 (1955).

The author establishes a Parseval formula:

$$M[|\tilde{\mu}|^2] = \sum_{\gamma \in G^*} |\mu\{\gamma\}|^2,$$

where μ is a bounded Radon measure on the dual G^* of a locally compact abelian group G, M denotes the mean value over G, and where $\tilde{\mu}$ is the Fourier transform of μ. The mean value used is that arising in the author's theory of weak almost periodic functions [Trans. Amer. Math. Soc. 67, 217–240 (1949); MR 12, 112]. The crux of the proof lies in showing that, with the mean value thus defined, one has $\mu\{0\} = M[\tilde{\mu}]$, a result easily established in the case of the real line owing to the special form of M there applicable. The formula is applied to give rapid proofs and extensions of some known results. *R. E. Edwards* (London).

Helson, Henry. Fourier transforms on perfect sets. Studia Math. 14 (1954), 209–213 (1955).

In continuance of work of I. E. Segal [Acta Sci. Math. Szeged 12, Pars B, 157–161 (1950); MR 12, 188, 1002], E. Hewitt [Proc. Amer. Math. Soc. 4, 663–670 (1953); Ark. Mat. 2, 571–574 (1954); MR 15, 119, 802] and the reviewer [Proc. Amer. Math. Soc. 5, 71–78 (1954); MR 15, 633], this paper deals with the restrictions, to a bounded perfect subset P of the real line, of the set of Fourier transforms of summable functions. Denote by $C(P)$ the set of continuous functions on P. The main theorem asserts that if $C(P)$ is covered entirely by such restrictions of Fourier transforms, then there exists no non-zero Radon measure, with support in P, whose Fourier transform tends to zero at infinity on the line. The method of proof is functional analytic. *R. E. Edwards* (London).

Mandelbrojt, Szolem. Quelques théorèmes sur les transformées de Fourier. C. R. Acad. Sci. Paris 240, 1393–1394 (1955).

Two theorems, claimed to be generalizations of Wiener's Tauberian theorems, are stated. In both, $K(x)$ is a function of $L(-\infty, \infty)$ whose Fourier transform $k(u)$ is non-zero for $u < -h'$ and $u > h$, where $h + h' \geqq 0$. I. If ess. lim. sup. $|F(x)| = M$, and $\int_{-\infty}^{\infty} K(x-y)F(y)dy = 0$, then there is an integral function $F_1(z)$ such that $F_1(x) = F(x)$ for almost all real x and $|F(z)| \leqq Me^{h'y}$ $(y \geqq 0)$, $|F(z)| \leqq Me^{-hy}$ $(y \leqq 0)$. II. If

$$\sum_{n=-\infty}^{\infty} \max_{x' \in [n, n+1]} |K(x)| < \infty, \quad \inf_{u>0} \sup_{a} u^{-1} \int_a^{a+u} |dF| = A,$$

and if $\int_{-\infty}^{\infty} K(x-y)dF(y) = 0$, there is an integral function $F_1(z)$, equal to $F(x)$ for almost all real x, and such that for any $\epsilon > 0$, and $|z| > r_\epsilon$, $|F(z)| \leqq 2(A+\epsilon)|z|e^{h'y}$ $(y \geqq 0)$, and for any $v > 0$, for $|x| > a_{\epsilon, v}$, $|y| < v$, we have

$$|F(z)| \leqq (A+\epsilon)|z|e^{h'y} \quad (y \geqq 0),$$

while for $y \leqq 0$ similar inequalities, with h' replaced by $-h$ in the exponential, are valid. *J. L. B. Cooper.*

Bojanić, R., et Tomić, M. Sur l'ordre de grandeur de la transformée de sinus de Fourier. Acad. Serbe Sci. Publ. Inst. Math. 7, 41–46 (1954).

Let $f(x)$ be a function decreasing monotonically to 0 as $x \to \infty$ such that, for some k, $0 < k < 2$, $x^k f(x)$ increases monotonically to ∞, and such that $F(t) = \int_0^\infty f(x) \sin tx\, dx$ exists. Then for sufficiently small t

$$At^{-1}f(t^{-1}) < F(t) < Bt^{-1}f(\pi t^{-1}),$$

for constants A and B (B depending on k). If, for some α, $0 < \alpha < k$ and $x^\alpha f(x)$ decreases to 0 monotonically, then the nequalities hold for all t. *J. L. B. Cooper* (Cardiff).

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Artémiadis, Nicolas K. Deux théorèmes sur les fonctions appartenant à la classe $L_1(-\infty, \infty)$. C. R. Acad. Sci. Paris 240, 1500–1502 (1955).

If $f(x) = 0$ $(x < 0)$ and $f \in L(-\infty, \infty)$, these theorems give an upper bound for $|f(x)|$ when the Fourier transform of $f(x)$ (complex or sine) has certain properties. The first theorem shows that if the complex Fourier transform $\phi(\alpha)$ of such a function has non-negative real part for all α, if $f(0) > 0$, and if $f(x)$ is reasonably well behaved at $x = x_0$ and $x = +0$, then $|f(x_0)| \leqq f(0)$. The second states that if the Fourier sine transform $\psi(\alpha)$ of such and $f(x)$ is not zero for any $\alpha > 0$, if $\int_{-\infty}^{\infty} |f(x)/x| dx$ exists, and if

$$\int_{-\infty}^{\infty} f(x)x^{-1}dx > 0, \quad \text{then} \quad |f(x)| < 2\pi^{-1}\int_0^\infty f(x)x^{-1}dx.$$

F. Goodspeed (Vancouver, B. C.).

Ovseevič, I. A., and Yaglom, A. M. Monotonic transfer processes in homogeneous long lines. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1954, no. 7, 13–20 (1954). (Russian)

The authors are concerned with the function $f(x, t)$ which for any fixed x is monotone in t and differentiable. Let

$f(x,+\infty)=1$ and $f(x,t)=0$, $t\leqq 0$. Let $f'(x,t)=\partial f/\partial t$ and let $F(x,\omega)$ be the Fourier transform $\int_0^\infty f'e^{-i\omega t}dt$. Then in the present application $\ln F(x,\omega)=x\ln F(1,\omega)$. Thus it suffices to deal with the case $f(1,t)$ to determine $F(1,\omega)$. Since $f(1,t)$ is a distribution function, recourse is had to probability theory to get information about $F(1,\omega)$. The vanishing of $f(1,t)$ for $t\leqq 0$ further restricts F, which finally takes the form

$$-\ln F(1,\omega)=c|\omega|^\alpha\left(1+j\operatorname{tg}\frac{\pi\alpha}{2}\frac{\omega}{|\omega|}\right)+j\tau\omega,$$

where $c\geqq 0$, $0<\alpha<1$, $\tau\geqq 0$. Several special cases are then worked out. *N. Levinson* (Cambridge, Mass.).

Pollard, Harry. The Poisson transform. Trans. Amer. Math. Soc. **78**, 541–550 (1955).

The Poisson transform is defined by the equation

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+(x-t)^2}d\alpha(t).$$

α is of bounded variation in each finite interval, and the integral is interpreted as $\int_{-\infty}^{+\infty}=\lim_{R\to\infty,\,S\to\infty}\int_{-S}^{R}$.

The author proves the following inversion formulas:

$$\tfrac{1}{2}[\alpha(x+)+\alpha(x-)]-\tfrac{1}{2}[\alpha(0+)+\alpha(0-)]=\lim_{t\to 1-}\int_0^x (T_t f(u))du,$$

and for almost all x

$$\Phi(x)=\lim_{t\to 1-}\int_0^x (T_t f(u))du,$$

where

$$\Phi(x)=(\cos D)f(x)+\frac{\sin D}{D}\tilde{f}'(x)$$

($\tilde{f}$ the Hilbert transform of f),

$$\hat{f}(x)=-\frac{1}{\pi}\int_0^\infty u^{-2}[f(x+u)-2f(u)+f(x-u)]du,$$

$$(\cos tD)f(x)=\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}f^{(2k)}(x),$$

$$\frac{\sin tD}{D}f(x)=\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}f^{(2k)}(x),$$

$$T_t f(x)=(\cos tD)f(x)+\frac{\sin tD}{D}\hat{f}(x).$$

W. Saxer (Zurich).

Polynomials, Polynomial Approximations

Walsh, J. L. A generalization of Jensen's theorem on the zeros of the derivative of a polynomial. Amer. Math. Monthly **62**, 91–93 (1955).

Soit $p(z)$ un polynome réel dont les zéros ont tous leur partie réelle dans l'intervalle $\alpha\leqq x\leqq\beta$, et soit γ extérieur à cet intervalle. Alors les zéros non réels de $p'(z)$ sont contenus dans les intérieurs fermés des cercles $\Gamma(z_k)$, passant par z_k, $\bar{z}_k$, et tangents à γz_k, où z_k, $\bar{z}_k$ désignent les couples de zéros non réels de $p(z)$. *J. Lelong* (Lille).

Palamà, Giuseppe. Limitazioni di taluni polinomi e in particolare di quelli di Laguerre. Boll. Un. Mat. Ital. (3) **10**, 47–51 (1955).

Bounds for a certain class of polynomials, which includes the Laguerre polynomials, are obtained. The method here consists of the development of the polynomial in a continued fraction. The results are applied to Laguerre polynomials.

E. Frank (Chicago, Ill.).

Nassif, M. Note on the Bessel polynomials. Trans. Amer. Math. Soc. **77**, 408–412 (1954).

This note contains two principal results concerning Bessel polynomials [H. L. Krall and O. Frink, same Trans. **65**, 100–115 (1949); MR **10**, 453]. The first is an expansion theorem based on methods developed by J. M. Whittaker and his students [J. M. Whittaker, Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; MR **11**, 344]. It is shown that if $f(z)$ is analytic in the region $|z-a|\leqq R$, then the series $\sum_0^\infty c_n p_n(z-a)$ converges uniformly to $f(z)$ in the same region, where $p_n(z)$ is the Bessel polynomial, and the coefficients c_n are given by $c_n=2^n(2n+1)\sum_0^\infty(-2)^k f^{(n+k)}(a)/k!(2n+k+1)!$. The second result is that all the zeros of the Bessel polynomial $p_n(z)$ for $n>1$ lie on or within the circle $|z|=(n-1)^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}$. This is an improvement on the result of E. Grosswald [Trans. Amer. Math. Soc. **71**, 197–210 (1951); MR **14**, 747], who showed that all the zeros for $n>1$ lie in the unit circle.

O. Frink (University Park, Pa.).

Singer, Ivan. Sur la meilleure approximation des fonctions continues par des combinaisons linéaires de fonctions données. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **6**, 465–475 (1954). (Romanian. Russian and French summaries)

The author generalizes some properties of best approximation by polynomials to approximation by systems of linearly independent continuous functions $\{\phi_k(x)\}_{k=0}^{n+1}$, forming a Chebyshev system: i.e., every nontrivial linear combination of m of the functions has at most $m-1$ roots on the interval under consideration. For convenience in notation the author considers the best uniform approximation $E_n(f)$ by an $(n+1)$-term linear combination of the $n+2$ given functions. Generalized divided differences are defined by putting

$$S(x_1,x_2,\cdots,x_{k+1})=\det\phi_i(x_{j+1})\quad(i,j=0,1,\cdots,k),$$

$$\{x_1,x_2,\cdots,x_{k+2};f\}=\frac{\begin{vmatrix}\phi_0(x_1)&\cdots&\phi_k(x_1)&f(x_1)\\ \cdots&\cdots&\cdots&\cdots\\ \phi_0(x_{k+1})&\cdots&\phi_k(x_{k+1})&f(x_{k+1})\end{vmatrix}}{S(x_1,\cdots,x_{k+1})}.$$

Then (1) if $|\{x_1,\cdots,x_{n+2};f\}|<|\{x_1,\cdots,x_{n+2};g\}|$ for all systems $\{x_k\}$, it follows that $E_n(f)<E_n(g)$. Now assume that the ϕ_k are n times differentiable and the Wronksians $W(\phi_0,\cdots,\phi_k)$ of any $k+1$ of them are all different from zero. If we define generalized interpolation polynomials based on ϕ_k in the obvious way, and $R_n(f)$ is the remainder in the interpolation formula for f, then we have that

$$(2)\qquad |W(\phi_0,\cdots,\phi_n,f)|<|W(\phi_0,\cdots,\phi_n,g)|$$

implies $|R_n(f)|<|R_n(g)|$. If $E_n^{(s)}(f)$ denotes the best sth power approximation to f, then (3) the hypothesis of (2) implies $E_n^{(s)}(f)<E_n^{(s)}(g)$. Finally the author makes some remarks on the connections between best approximation and convex functions of order n. *R. P. Boas, Jr.*

Gol'dštein, E. G. On best approximations of harmonic functions by harmonic polynomials. Dokl. Akad. Nauk SSSR (N.S.) **101**, 5–8 (1955). (Russian)

Let $f(P)$ be a continuous function defined in a closed set F of three-dimensional Euclidean space, harmonic at all interior points of F. Put

$$E_n(f)=\inf_{Q}\sup_{P\in F}|f(P)-Q(P)|,$$

where $Q(P)$ runs through all harmonic polynomials of degree $\leqq n$ in the cartesian coordinates of P. The author gives upper bounds for $E_n(f)$ under rather general assumptions about F. A typical result which is mentioned as a corollary of his theorems is the following: Let F be the closure of a bounded domain whose boundary has a continuously turning tangent plane. Let $f(P)$ be continuous with modulus of continuity $\omega(\delta)$. Given $\epsilon>0$ there is a constant $c(\epsilon)$ such that $E_n(f)\leqq c(\epsilon)\omega(n^{-1+\epsilon})$. The method of proof utilizes pole shifts, following an idea due to M. Keldyš [see S. N. Mergelyan, Uspehi Mat. Nauk (N.S.) **8**, no. 4(56), 3–63 (1953); MR **15**, 411]. *W. H. J. Fuchs* (Ithaca, N. Y.).

Special Functions

Monna, A. F. The introduction of the logarithm. Euclides, Groningen **30**, 88–96 (1955). (Dutch)

Engaging in "elementary mathematics from an advanced standpoint", the author defines and studies the logarithm, starting with Haar measure μ on the multiplicative group of positive numbers. For instance, if $a>1$, μ is normalized such that $\mu((1,a))=1$, and $\log_a x$ is then defined to be $\mu((1,x))$ if $x>1$, 0 if $x=1$, and $-\mu((1,1/x))$ if $0<x<1$.

E. Michael (Seattle, Wash.).

Zonneveld, J. A., and Berghuis, J. The asymptotic expansion of a special function and some relations with Bessel functions. Math. Centrum Amsterdam. Rekenafdeling no. **MR 18**, 8 pp. (1955).

The special function considered is

$$I(t,x)=2\pi^{-1}\int^{\infty} f(t,u)g(x,u)du,$$

where $f(t,u)=u^{-2}[1-\exp(-u^2t)]$ and

$$g(x,u)=[J_1(u)Y_0(xu)-Y_1(u)J_0(xu)][J_1^2(u)+Y_1^2(u)]^{-1}.$$

The asymptotic expansion of the Laplace transform of I is found, and the Mellin transformation is then used to obtain the expansion of $I(t,x)$ for small $t>0$ when $x\geqq 1$. This expansion is also derived by applying a theorem of Berghuis [Math. Centrum Amsterdam. Rekenafdeling **MR 11** (1953); MR **15**, 951]. Certain special integrals related to $I(t,x)$ are shown to equal zero, e.g.,

$$\int_0^{\infty}[g(x,u)-x^{-1/2}\cos(x-1)u]\,du=0.$$

N. D. Kazarinoff (Lafayette, Ind.).

Huet, Denise. Sur la confluence des fonctions de Bessel généralisées. C. R. Acad. Sci. Paris **240**, 1297–1298 (1955).

The functions to which the title refers were introduced by Delsarte [Premier colloque sur les équations aux dérivées partielles, Louvain, 1953, Thone, Liège, 1954, pp. 35–62; MR **15**, 870]. They are solutions of a certain system of partial differential equations and involve a parameter q analogous to the order of the classical Bessel functions. The author remarks that for certain values of q these functions are not all distinct. The system then has logarithmic solutions the form of which he determines.

N. D. Kazarinoff (Lafayette, Ind.).

Preuss, H. Zur Behandlung der Zweizentren-Wechselwirkungsintegrale. Z. Naturf. **10a**, 211–215 (1955).

Harmonic Functions, Potential Theory

Hitotumatu, Sin. Note on the Riesz decomposition of a subharmonic function. Comment. Math. Univ. St. Paul. **3**, 69–94 (1955).

L'A. établit la décomposition de Riesz (avec le potentiel de Green dans un domaine de R^n) en s'inspirant de la méthode des distributions de Schwartz, mais intentionellement sans utiliser cette théorie. Il rappelle les principes de l'intégrale de Radon, présentée au point de vue fonctionnel et se base naturellement sur le critère d'harmonicité de Weyl qui revient à dire qu'une fonction continue dont le laplacien de Schwartz est nul est harmonique. *M. Brelot.*

Tsuji, Masatsugu. On a positive harmonic function in a half-plane. J. Math. Soc. Japan **7**, 76–78 (1955).

L'A. montre que si u harmonique >0 pour $x>0$ est bornée sur un arc aboutissant à l'origine O et situé dans un angle axé sur Ox et d'ouverture $<\pi$, u est bornée dans tout angle analogue, au voisinage de l'origine. La démonstration est basée sur la représentation intégrale des fonctions harmoniques >0. *M. Brelot* (Paris).

Ozawa, Mitsuru. Some classes of positive solutions of $\Delta u=Pu$ on Riemann surfaces. I. Kōdai Math. Sem. Rep. **1954**, 121–126 (1954).

For non-compact Riemann surfaces F satisfying certain restrictive conditions correspondences are established between certain classes of positive harmonic functions and certain classes of positive solutions of $\Delta u=Pu$, where P is a C' non-negative density vanishing on a set clustering at no point of F. The work of this note has connections with the work of L. Myrberg [Ann. Acad. Sci. Fenn. Ser. A. I. no. **170** (1954); MR **16**, 34]. *M. Heins.*

Blumer, Hans. Beziehungen zwischen speziellen linearen Integralgleichungen erster und zweiter Art und Lösung des Dirichletschen Problems durch das Potential einer einfachen Schicht. Comment. Math. Helv. **28**, 197–224 (1954).

Soit G un domaine borné de R^3, limité par une surface assez régulière Ω. Si on cherche à résoudre le problème de Dirichlet classique pour G avec donnée-frontière f sur Ω au moyen du potentiel d'une simple couche g sur Ω, on est conduit à l'équation intégrale de première espèce

$$(1)\qquad f(p)=\int_{\Omega} r_{pq}^{-1}g(q)d\omega(q)$$

qui, comme on sait, n'a pas de solution pour toute f continue. L'auteur montre que si $f\in C^2$ il existe une solution g continue [résultat connu si G est un domaine plan: cf. G. Bertrand, Bull. Sci. Math. (2) **47**, 282–288, 298–307 (1923)] en formant dans ce cas une équation de Fredholm satisfaite

par g. La méthode, valable pour des équations plus générales que (1), consiste à appliquer aux deux membres de (1) un opérateur intégro-différentiel I^s analogue à celui de M. Riesz [Acta Math. 81, 1–223 (1949); MR 10, 713]. La définition et les propriétés de I^s nécessitent une étude préliminaire de l'équation de Laplace-Beltrami sur Ω en relation avec un travail récent de Minakshisundaram et Pleijel [Canad. J. of Math. 1, 242–256 (1949); MR 11, 108]. La solution g est explicitée dans le cas de la sphère.

J. Deny (Princeton, N. J.).

Segal, B. I. Spatial problems of potential theory for cylindrical regions. Izv. Akad. Nauk SSSR. Ser. Mat. 16, 59–74 (1952). (Russian)

Vorgelegt sei als Modell für eine unterirdische Wasserader ein halbunendlicher Kreiszylinder (Zylinderachse ist die z-Achse), in dessen Grundfläche, die als undurchlässig angenommen wird, sich n symmetrisch gelegene Vertiefungen ("Bohrlöcher") befinden, deren Abmessungen gegenüber ihrem Abstand vom Mittelpunkt der Grundfläche klein sind. Die Bestimmung des Geschwindigkeitspotentials u innerhalb des Zylinders führt auf eine potentialtheoretische Randwertaufgabe mit den folgenden Forderungen: Auf dem Zylindermantel ist $u=0$, auf dem nichtangebohrten Teil $\partial u/\partial z=0$, auf der Oberfläche der Bohrlöcher $u=u_1=\text{const.}$, $u(\infty)=0$. Zur Lösung werden in den Mittelpunkten der Bohrlöcher auf der Grundfläche geeignete Senken angebracht, deren Potential mit Hilfe von Reihenentwicklungen nach Besselfunktionen ausdrückbar ist. Beschränkt man sich, was im Hinblick auf die Anwendungen möglich ist, nur auf die Anfangsglieder, so gelingt es, die Lösung des vorgelegten Problems schrittweise aufzubauen. Betrachtet werden speziell Bohrlöcher in Gestalt von Halbkugeln und Zylindern, ferner auch der Fall einer endlichen zylindrischen Wasserader. Die Ergebnisse numerischer Rechnungen finden sich in einigen Tabellen. *K. Maruhn* (Dresden).

Teleman, S. Une propriété des fonctions du développement d'Almansi d'une fonction polyharmonique. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. 5, 385–391 (1953). (Romanian. Russian and French summaries)

It is shown that, in the Almansi development

$$F(M)=F_1(M)+r^2F_2(M)+\cdots+r^{2(p-1)}F_p(M)$$

of a polyharmonic function $F(M)$ of order p, where $r=\overline{MM_0}$ with M_0 fixed in a given development, the coefficient F_i is polyharmonic of order $p-i+1$ with respect to M_0.

E. F. Beckenbach (Los Angeles, Calif.).

Differential Equations

Kimura, Toshifusa. Sur une généralisation d'un théorème de Malmquist. II. Comment. Math. Univ. St. Paul. 3, 97–107 (1955).

[For part I see same Comment. 2, 23–28 (1953); MR 15, 311.] The author considers the equation (1) $dy/dx=R(x, y)$, where R is rational in y, with coefficients analytic at the origin. If (1) has an integral with the origin as an e.s.p. (essentially singular point), the equation has form (2) $x^{\sigma+1}dy/dx=P(x, y)/Q(x, y)$, where P, Q are polynomials in y, without common factor and not containing x as a factor; $\sigma\geqq 0$ is an integer. Results such as the following are proved. If there exists at least one zero γ of $Q(0, y)$ of multiplicity ν, $\nu\geqq\mu$ $(\geqq 0)$, where μ is the multiplicity of the zero γ of $P(0, y)$, then in an arbitrary vicinity of the e.s.p. $x=0$ there is an infinity of movable critical points. The forms are given, to which (2) can be reduced by a homographic transformation when (2) is not of Riccati type and if (2) has an integral with origin for an e.s.p. in a vicinity of which this integral has no movable critical points. *W. J. Trjitzinsky.*

Andronov, A. A., and Leontovič, E. A. Generation of limit cycles from a structurally unstable focus or from a center and from a structurally unstable limit cycle. Dokl. Akad. Nauk SSSR (N.S.) 99, 885–888 (1954). (Russian)

Consider two real systems

$$(1)\qquad \dot{x}=P(x, y),\qquad \dot{y}=Q(x, y),$$
$$(2)\qquad \dot{x}=P(x, y)+p(x, y),\quad \dot{y}=Q(x, y)+q(x, y),$$

where all the quantities are of class C^N or all analytical in some region g (or its closure) containing the origin, which is a focus for (1) and a critical point for (2). The system (2) is said to be (s, δ) close to (1), if p, q and all their partials up to order s are $<\delta$ in absolute value (understood throughout in g). In polar coordinates (1) and (2) become

$$(3a)\qquad \frac{dr}{d\theta}=R(r, \theta);\qquad (3b)\qquad \frac{dr}{d\theta}=R(r, \theta)+R^*(r, \theta)$$

with behavior (as to class) analogous to that of (1) and (2). The system (3a) has a general solution $r(\theta, \rho)$ where $\rho=r(0, \rho)$. Let $\psi(\rho)=r(2\pi, \rho)-\rho$. Then the origin is a focus of multiplicity k whenever $\psi^{(i)}(0)=0$, $i=1, 2, \cdots, 2k$. Theorem 1. If the origin is a focus of multiplicity k, $N=2k+1$, for (1), then: (a) there exist ϵ_0, $\delta_0>0$ such that any system (2) (N, δ_0) close to (1) has at most k limit-cycles within ϵ_0 of the origin; (b) for any $\epsilon<\epsilon_0$ and $\delta<\delta_0$, there exist systems (2) (N, δ) close to (1) and with k limit cycles within ϵ of the origin. Theorem 2: Analogous statement with "focus" replaced by "limit-cycle". *S. Lefschetz.*

Putnam, C. R. A note on correlation functions and stability in dynamical systems. Proc. Amer. Math. Soc. 5, 696–699 (1954).

Given a system of differential equations $x'=f(x)$ where $x=(x_1, \cdots, x_n)$, let $f(x)=(f_1, \cdots, f_n)$ be of class C' with

$$\operatorname{div} f=\partial f_1/\partial x_1+\cdots+\partial f_n/\partial x_n=0$$

throughout an invariant set Ω which is the closure of a bounded open set in n-space. Let $P_t=x(t)$ denote a solution of the system passing through $P=x(0)$. Such a path $x(t)$ is called stable (with respect to Ω) if for every $\epsilon>0$ there exists a $\delta>0$ such that $|x(t)-y(t)|<\epsilon$ for $-\infty<t<\infty$ whenever $y(t)$ is a solution of the system lying in Ω with $|x(0)-y(0)|<\delta$. Let v be the ordinary Lebesgue volume measure in Ω.

If $f=f(P)$ is of class (L^2) on Ω, then for almost all P the correlation function

$$c_P(s)=\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}f(P_{t+s})f(P_t)dt$$

exists and is a continuous function of s on $-\infty<s<\infty$ and there is a unique monotone function $\sigma_P(\lambda)$ such that $\sigma_P(-\infty)=0$, $\sigma_P(\lambda-0)=\sigma_P(\lambda)$ and $C_P(s)=\int_{-\infty}^{\infty}e^{i\lambda s}d\sigma_P(\lambda)$. [cf. Wiener and Wintner, Amer. J. Math. 63, 794–824 (1941); MR 4, 15].

The author proves that if P_t^* is a stable path for which $f(P_t^*)$ is almost periodic (B^2) [cf. Besicovitch, Almost periodic functions, 1932, Cambridge, p. 78], then the set

of frequencies of $f(P_t^*)$, i.e. the point spectrum of $\sigma_{P^*}(\lambda)$, is contained in the point spectrum of $\Sigma(\lambda)=\int_\Omega \sigma_P(\lambda)dv$, that is in the set of frequencies of

$$F(s)=\int f(P_s)f(P)dv=\int_{-\infty}^{\infty} e^{i\lambda s}d\Sigma(\lambda).$$

If also $f(P_t^*)$ is uniformly almost periodic in the sense of Bohr and $F(s)\to 0$ as $|s|\to\infty$, then $f(P_t^*)\equiv 0$ for $-\infty<t<\infty$.

Y. N. Dowker (London).

Amerio, Luigi. Varietà analitiche chiuse trasformate in sè dai sistemi differenziali periodici. Ann. Mat. Pura Appl. (4) **37**, 219–248 (1954).

Consider a system $\dot{x}=X(t,x)$, x, X being (real) n-vectors, X periodic in t with period T, analytic in x in a certain domain; the system may then also be considered as a system in a complex vector space (t remaining real). For $t=\bar{t}$, let λ_0 be a closed analytic $(n-1)$-manifold in the real x-space (given by a finite number of representations $x=\psi(\tau)$, τ being a real $(n-1)$-vector) bounding a domain D_0, such that the following assumptions are satisfied: α) the integral curves with initial points $(\bar{x},\bar{t})$, $\bar{x}\in\lambda_0$, exist for $t\geqq\bar{t}$; they generate an n-manifold Λ_0 in the (real) (x,t)-space, whose sections $t=$const. are uniformly bounded; the intersections D_k of the domain Δ bounded by $\Lambda_0\cup D_0$ and the planes $t=\bar{t}+kT$ satisfy $D_{k+1}\subset D_k$; β) if $\lambda_{0'}$ is the extension of λ_0 to the complex space (i.e. the manifold defined by $x=\psi(\tau)$, τ complex $(n-1)$-vector with sufficiently small imaginary parts), the (complex) integral curves with initial points $(\bar{x}',\bar{t})$, $\bar{x}'\in\lambda_{0'}$, are uniformly bounded (in the complex x-space). Then it follows from the consideration of normal families of analytic functions that there is a closed analytic (real) p-manifold λ, $p\leqq n-1$, such that $\lambda=\phi(\lambda)$, where ϕ is the homeomorphism of the x-space induced by the system $\dot{x}=X$. The integral curves with initial points $(\bar{x},\bar{t})$, $\bar{x}\in\lambda$, are almost periodic; moreover, if $n=2$ and one of such curves is periodic, then all of them are periodic (the period being a multiple of T).

J. L. Massera (Montevideo).

Montaldo, Oscar. Sul sistema di due equazioni differenziali ordinarie del secondo ordine omogenee nelle derivate prime. Rend. Sem. Fac. Sci. Univ. Cagliari **24**, 1–9 (1954).

The family of trajectories of a particle moving in accordance with the equations of motion $\ddot{x}=\varphi(x,y)$, $\ddot{y}=\psi(x,y)$ is defined by the equation

$$\text{(A)}\quad (\psi-y'\varphi)y'''=[\psi_x+(\psi_y-\varphi_x)y'-\varphi_y y'^2]y''-3\varphi y'^2,$$

where the primes indicate differentiation with respect to x. Kasner [Differential-geometric aspects of dynamics, Amer. Math. Soc. Colloq. Publ., v. 3₂, New York, 1913] gave a set of five properties which characterizes families of curves defined by equations of the form (A). In the present paper the author discusses systems of equations of motion of the form $\ddot{x}=\Phi_n(x,y,\dot{x},\dot{y})$, $\ddot{y}=\Psi_n(x,y,\dot{x},\dot{y})$, where Φ_n and Ψ_n are forms of degree n ($\neq 2$) in $\dot{x}$ and $\dot{y}$, which also lead to families of trajectories defined by equations of the form (A) (and hence characterized by Kasner's properties). The most general such system of equations of motion is rather complicated, but a simple example is the following:

$$\ddot{x}=\varphi(x,y)\dot{x}^n,\quad \ddot{y}=\psi(x,y)\dot{x}^n+\tfrac{1}{2}n\varphi(x,y)\dot{x}^{n-1}\dot{y}.$$

L. A. MacColl (New York, N. Y.).

Tatarkiewicz, Krzysztof. Sur l'allure asymptotique des solutions de l'équation différentielle du second ordre. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 19–81 (1954). (Polish and Russian summaries)

The author applies Ważewski's topological method to the discussion of boundedness and asymptotic behavior of the solutions of the equation (1) $\ddot{x}-2a(t)\dot{x}-b(t)x=f(t)$, where $a(t)$, $b(t)$, $f(t)$ are continuous in $0\leqq t<+\infty$. If for every t, $h_1(t)$, $h_2(t)$, $0\leqq t<+\infty$, are the roots of the algebraic equation $h^2-2a(t)h-b(t)=0$, various conditions are discussed under which all the functions $h_j(t)$, $j=1,2$, have real parts $\Re[h_j(t)]$ of constant sign in $[0,+\infty)$. Let $K=0,1,2$ be the number of the functions $\Re[h_j(t)]$, $j=1,2$, which are positive. For instance, the following conditions are considered: (1) $|a(t)|<A_2$, $0<B_1<b(t)$ for some constants $A_2\geqq 1$, $B_1>0$ and all t (thus $K=1$); (2) $0<a(t)<A_2$, $0<B_1<-b(t)<a^2(t)-b$ for some positive constants A_2, B_1, b (thus $K=2$), and $a^1(t)$, $b^1(t)$ are continuous and

$$m\tau(t)\left|h_j(t)/b(t)\right|\geqq\left|[a(t)/b(t)]'\right|,$$
$$\tau(t)\left|h_j(t)/b(t)\right|\geqq 2(-1)^j\left|[\tau(t)/b(t)]'\right|,\quad j=1,2,$$

for some constant $0<m<1$, where $\tau^2(t)=a^2(t)+b(t)$. Under either condition (1) or (2) above, if $f(t)$ is bounded, then there is a K-parameter family of solutions of (1) which are bounded in $[0,+\infty)$ with their first derivatives. All remaining solutions are not bounded and their behavior is further characterized. [Consistent use of the roots $h_j(t)$ (functions of t) was made by the reviewer [Ann. Scuola Norm. Super. Pisa (2) **9**, 163–186 (1940); MR **3**, 41] under the hypotheses $a(t)$, $b(t)$ of bounded variation in $[0,+\infty)$ (not necessarily continuous), and $\Re[h_j(t)]\geqq 0$ for nth order differential equations [also successively by N. Levinson, Duke Math. J. **15**, 111–126 (1948); MR **9**, 509]. The reviewer agrees with the author that conditions of differentiability of $a(t)$, $b(t)$ should not be as essential as they appear, for instance in condition (2).]

L. Cesari (Lafayette, Ind.).

Yakubovič, V. A. Estimates of characteristic exponents of a system of linear differential equations with periodic coefficients. Prikl. Mat. Meh. **18**, 533–546 (1954). (Russian)

Consider (1) $\dot{x}=A(t)\cdot x$, A periodic of period ω; let $G(t)$ be any hermitian, periodic, differentiable, positive definite matrix. Then, if λ_1, λ_2 are the characteristic exponents of (1) having smallest and largest real parts,

$$(1/2\omega)\int_0^\omega q_1 dt\leqq \operatorname{Re}\lambda_1\leqq\operatorname{Re}\lambda_2\leqq(1/2\omega)\int_0^\omega q_2 dt,$$

where $q_1(t)$, $q_2(t)$ are the smallest and largest roots of $\det(Q-qG)=0$, $Q=\dot{G}+GA+A^*G$; these bounds are best possible. The author shows on several examples the usefulness of this result; in particular, equations $y''+P(t)y=0$, y scalar, and canonical second-order systems are considered. These special results are too complicated to be reproduced here.

J. L. Massera (Montevideo).

Yakubovič, V. A. Extension of Lyapunov's method of determining boundedness of solutions of the equation $y''+p(t)y=0$, $p(t+\omega)=p(t)$ to the case of a function $p(t)$ of variable sign. Prikl. Mat. Meh. **18**, 705–718 (1954). (Russian)

The stability of the solutions of $y''+p(t)y=0$, p periodic of period ω, can be ascertained by means of the characteristic constant $A=\frac{1}{2}[\varphi(\omega)+\psi'(\omega)]$, where φ, ψ are solutions satisfying $\varphi(0)=\psi'(0)=1$, $\varphi'(0)=\psi(0)=0$; if $|A|>1$ the solu-

tions are unstable, if $|A|<1$ stable, $|A|=1$ being a doubtful case. Lyapunov [Zap. Imp. Akad. Nauk. Fiz.-Mat. Otd. (8) 13, no. 2 (1902)] gave a constructive method to calculate A in the case $p\geqq 0$; the same problem is considered here when p changes sign. It is shown that $A=\sum(-1)^n A_n$, where the A_n are given by quadratures; the A_n are positive and, from a certain n_0 on, they decrease monotonically; except in the doubtful case the stability or instability may be decided by means of a finite number of operations. In particular, the following sufficient criteria are established: let $a^2=\sup\{-p(t)\}$; then, if $\int_0^\omega (p(t)+a^2)dt\leqq 2a\tanh(a\omega/2)$ the solutions are unstable; if

$$\int_0^\omega p(t)dt\geqq 0,\quad \int_0^\omega (p(t)+a^2)dt\leqq 2a\operatorname{cotanh}(a\omega/2),$$

the solutions are stable. *J. L. Massera* (Montevideo).

Dubošin, G. N. On integration of a system of linear equations of the second order by the method of A. M. Lyapunov. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. **24**, 109–121 (1954). (Russian)

The paper purports to solve

(1) $$\ddot{x}=Q(t)\dot{x}+P(t)x$$

(x an n vector, P and Q matrices) as follows. Replace (1) by

(2) $$\ddot{x}=\bar{Q}\dot{x}+\bar{P}x+\epsilon(Q-\bar{Q})\dot{x}+(P-\bar{P})x.$$

Solve (2) by series

(3) $$x=x^{(0)}+\epsilon x^{(1)}+\cdots$$

absolutely convergent in some interval around $\epsilon=0$; then make $\epsilon=1$, thus obtaining a solution of (1) for which many formulas are developed. (The reviewer fails to see that (3) converges for $\epsilon=1$.) *S. Lefschetz* (Mexico, D.F.).

Lefschetz, Solomon. Complete families of periodic solutions of differential equations. Comment. Math. Helv. **28**, 341–345 (1954).

Poincaré's method of small parameters for finding periodic solutions of real differential equations leads to a set of real analytic equations $f_i(x_1,\cdots,x_n;\mu)=0$, where $x_1,\cdots,x_n$ are the initial values of the periodic solution, and μ is the small parameter. The author gives an algorithm, based on Kronecker's elimination method, for finding those real solutions $x_j=x_j(\mu)$ which actually depend on μ.

G. E. H. Reuter (Manchester).

Stoker, J. J. On the stability of mechanical systems. Comm. Pure Appl. Math. **8**, 133–141 (1955).

This is an expository article describing, comparing, and criticizing various concepts of stability of a motion of a mechanical system. The difficulties attending the effective applications of the criteria for stability, and the stabilizing effect of small imposed oscillations, are discussed briefly.

L. A. MacColl (New York, N. Y.).

Hahn, Wolfgang. Über Zusammenhänge zwischen den graphischen und den algebraischen Stabilitätskriterien. Z. Angew. Math. Mech. **35**, 119 (1955).

Krasovskiĭ, N. N. On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **101**, 17–20 (1955). (Russian)

Preliminary announcement of results in Prikl. Mat. Meh. **18**, 513–532 (1954); MR **16**, 473. *S. Lefschetz.*

Orlov, A. A. On the integration of the differential equations of perturbed motion in rectangular coordinates by means of a small parameter. Moskov. Gos. Univ. Soobšč. Astr. Inst. no. **88–89**, 39–53 (1953). (Russian)

Consider the motion of a particle P under the Newtonian attraction from a fixed point O (the assumption that O is fixed is not essential but is made to simplify the problem) and subject to a perturbation force which depends on the position of P, the time t and a small parameter α. If the origin of a rectangular coordinate system is taken at O, the differential equations of motion of P can be easily written down. Call these equations (1).

Assume that for $\alpha=0$ the perturbation force vanishes. Then the equations (1) reduce to those for the problem of a particle attracted from a fixed center, say (2), and can be solved. Consider the case corresponding to the elliptic motion of P and call the solution (3). For $\alpha\neq 0$ Poincaré's small-parameter method is applied. This method, as is well-known, leads to the solution of an infinite sequence of systems of linear nonhomogeneous differential equations. Each such system determines the coefficients of equal powers of α in the expansions of the coordinates x, y, z of P. Consider the nth system and denote it by (4).

When the right-hand sides in (4) are put equal to zero, these equations reduce to the variational equations (5) of the unperturbed motion (2). The author first shows the construction of the general solution of these variational equations for an arbitrary elliptic solution (3) of the system (2). Afterwards, by application of the method of variation of constants, the solution of the system (4) is obtained for the case when the right-hand sides of these differential equations are different from zero.

Contrary to other methods for solving differential equations (1) of perturbed motion in rectangular coordinates no use is made of integrals or quasi-integrals of these equations.

E. Leimanis (Vancouver, B. C.).

Mitropol'skiĭ, Yu. A. Forced oscillations in nonlinear systems while passing through resonance. Inžen. Sb. **15**, 89–98 (1953). (Russian)

This is a study of the forced oscillations of

$$\frac{d}{dt}(m(\tau)\cdot\dot{x})+k(\tau)x=\epsilon F(\tau,\theta,x,\dot{x}),$$

where $\tau=\epsilon t$ and F is representable as a Fourier series in θ with coefficients polynomials in x and $\dot{x}$ and $\dot{\theta}=\nu(\tau)$ the frequency of F which varies with t. The variables m, k, ν are supposed to vary slowly relative to the period of the self-oscillations. The method of attack is essentially à la Krylov-Bogoliubov [Introduction to non-linear mechanics, Princeton, 1943; MR **4**, 142]. *S. Lefschetz.*

Kac, A. M. Forced oscillations of nonlinear systems with one degree of freedom and near to conservative ones. Prikl. Mat. Meh. **19**, 13–32 (1955). (Russian)

Drozdov, Yu. M. Forced oscillations of nonlinear systems with one degree of freedom and close to conservative ones (examples). Prikl. Mat. Meh. **19**, 33–40 (1955). (Russian)

The first paper was found among the manuscripts (in incomplete form) of the author after his death and prepared for publication by Drozdov and Lur'e. It discusses the periodic solutions of

(1) $$\ddot{x}+F(x)=\epsilon f(x,\dot{x},\epsilon,t),$$

where F is analytic in x and $f(x, u, \epsilon, t)$ in x, u, ϵ, and has period T in t. It starts with a solution x_0 of

$$\ddot{x}_0+F(x_0)=0. \tag{2}$$

Such a solution may have a period varying continuously in a certain interval. Let x_0 be a solution having the period T. Then (1) has a solution of the form

$$x=x_0+\epsilon x_1+\epsilon^2 x_2+\cdots. \tag{3}$$

Upon substituting in F and f one obtains

$$F=F_0+\epsilon F_1+\cdots, \quad f=f_0+\epsilon f_1+\cdots.$$

There results an infinite system of periodic differential equations for the x_n:

$$\ddot{x}_1+F'(x_0)x_1=f_0(x_1, \dot{x}_0, t)$$
$$\cdots \qquad \cdots$$
$$\ddot{x}_{n+1}+F'(x_0)x_{n+1}=g_n(x_0, \dot{x}_0, \cdots, x_n\dot{x}_n, t)$$
$$\cdots \qquad \cdots$$

whose periodic solutions can be obtained step by step and are discussed, together with their stability, by the author.

It should be noted that subharmonic solutions enter into play. If the true period of f is T/m, and the true period of x_0 is T/n, then the author calls the solution x an m/n solution.

The paper by Drozdov discusses three applications of the results of Kac. *S. Lefschetz* (Mexico, D. F.).

Zlámal, Miloš. Über die Stabilität der nichtlinearen erzwungenen Schwingungen. Čehoslovack. Mat. Ž. **4**(79), 95–103 (1954). (Russian summary)

Consider $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$, p periodic of period T. 1. If there are constants $\omega>c>0$, L_1, L_2 such that

$$\omega L_1+L_2<c(\omega^2-c^2)^{1/2}, \quad |f(x)-2c| \leqq L_1,$$
$$|\psi(x_1)-\psi(x_2)| \leqq L_2|x_1-x_2|,$$

where $\psi=g-\omega^2 x$, there is an asymptotically stable periodic solution. 2. The same assertion is true if $f=2c=$const. and $m(x_1-x_2) \leqq g(x_1)-g(x_2) \leqq M(x_1-x_2)$, $0<m \leqq M<2c^2$.

J. L. Massera (Montevideo).

Caprioli, Luigi. Sulle soluzioni periodiche di una equazione fortemente non-lineare. Boll. Un. Mat. Ital. (3) **9**, 271–280 (1954).

Standard methods are used to show that

$$\ddot{x}+\omega\epsilon^{-1}f(\dot{x})+\omega^2 x=0,$$

with $f(x)=x^5-kx^3+x$ ($k>0$), has two non-trivial periodic solutions ("hard excitation") for certain values of k: $k^2>5$ if ϵ is small, $k^2>40/9$ if ϵ is large. *G. E. H. Reuter.*

Furuya, Shigeru. Periodic solutions of the van der Pol-Mathieu equation. Comment. Math. Univ. St. Paul. **3**, 109–113 (1955).

The author studies the periodic solutions of the equation

$$\ddot{x}+e(x^2-1)\dot{x}+[1+(a-cx^2)\cos 2t]x=0$$

for small values of e, a and c by means of the perturbation method. This equation was studied earlier by Minorsky [J. Franklin Inst. **256**, 147–165 (1953); MR **15**, 427], but, according to the present author, Minorsky's results are somewhat incorrect. *C. E. Langenhop* (Ames, Ia.).

Minorsky, Nicolas. Sur l'espace paramétrique de l'équation de M. Liénard. C. R. Acad. Sci. Paris **240**, 1508–1509 (1955).

The author considers the differential equation

$$\ddot{x}+(a+cx^2+ex^4)\dot{x}+x=0,$$

where a, c, e are small parameters having comparable magnitudes, and $e>0$. Using the stroboscopic method, he studies the dependence of the topological structure of the family of trajectories in the phase plane upon the values of the parameters. It is found that the (a, c, e)-space is composed of regions of three kinds, called, respectively, acyclic, monocyclic, and bicyclic. If the parametric point (a, c, e) is in an acyclic region, there is a stable singular point and no closed trajectory; if (a, c, e) is in a monocyclic region, there is one unstable singular point and one (stable) closed trajectory; if (a, c, e) is in a bicyclic region, there is a stable singular point, an unstable closed trajectory, and a stable closed trajectory. *L. A. MacColl* (New York, N. Y.).

Belyustina, L. N. On the stability of the operating regime of a salient-pole synchronous motor. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 10, 131–140 (1954). (Russian)

The author considers the differential equation

$$\ddot{\theta}+k(1-b\cos 2\theta)\dot{\theta}+\sin\theta+r\sin 2\theta=T$$

where k, b, r and T are constants and $\dot{\theta}=d\theta/dt$. Setting $\dot{\theta}=z$, the equation is replaced in the usual way by a pair of first-order autonomous equations in the (θ, z) phase-plane which are studied by considering the nature of the singular points in the phase-plane. *N. Levinson* (Cambridge, Mass.).

***Vallese, L. M. On the synthesis of nonlinear systems.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 201–214. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

General considerations on the synthesis of nonlinear systems are made with a few remarks which may be useful to solve such problems. *J. L. Massera* (Montevideo).

Poli, L. Polynomes d'Hermite et équations différentielles. Mathesis **63**, 319–325 (1954).

The author is concerned with integrating differential equations of the form

$$y^{(n)}+\binom{n}{1}y^{(n-1)}P_1(x)+\binom{n}{2}y^{(n-2)}P_2(x)+\cdots+yP_n(x)=0,$$

where the coefficients $P_\nu(x)$ are Appell polynomials, i.e. satisfy $dP_\nu(x)/dx=\nu P_{\nu-1}(x)$. A solution is given by

$$y=\exp\{rx-\tfrac{1}{2}x^2\},$$

where r satisfies a polynomial equation of degree n involving the coefficients of the nth Hermite polynomial and of $P_n(x)$. The general solution is obtained by superposition if all the roots of this equation are distinct. Various special choices for the coefficients are discussed, including Hermite polynomials themselves and various Appell systems generalizing them.

In case the coefficients are not an Appell system, but are essentially derivatives of a given function, the differential equation can be put in the form $(y(x)z(x))^{(n)}=0$ by introducing a suitable integrating factor. Here $z(x)$ is given in terms of the coefficients $P_\nu(x)$. This is carried out in case

$$P_\nu(x)=\exp\left\{\frac{x^p}{p}\right\}\frac{d^{\nu+\gamma}}{dx^{\nu+\gamma}}\left(\exp\left\{-\frac{x^p}{p}\right\}\right)(-1)^{\nu+\gamma}.$$

As an application the formula

$$H_n(r)H_j(r)=\sum_{\nu=0}^{\min(j,n)}\binom{n}{\nu}\binom{n}{j}\nu!H_{n+j-2\nu}(r)$$

is derived where $H_n(x)=\exp\{\frac{1}{2}x^2\}d^n\exp\{-\frac{1}{2}x^2\}/dx^n$.

A. B. Novikoff (Baltimore, Md.).

Weinberger, H. F. An extension of the classical Sturm-Liouville theory. Duke Math. J. **22**, 1–14 (1955).

Let A and B denote two real self-adjoint boundary problems involving the same differential equation

$$p_{2N}(x)d^{2N}u/dx^{2N}+\cdots+p_0(x)u-(-1)^N\lambda q(x)u=0,$$

and respective boundary conditions at a specified set of values $x_1<x_2<\cdots<x_k$. Under the assumption that the Green's function, eigenfunctions and eigenvalues of problem A are known, and that the boundary conditions of B differ from those of A by R conditions, it is shown that for problem B the eigenvalues, with their proper multiplicities, are determined by a single transcendental equation that is the determinant of a certain $R\times R$ matrix. The development of the author leads to some separation theorems for the eigenvalues of the two problems. For $R=1$ there is given a criterion involving the Green's function of one of the problems and the altered boundary condition. In addition there are derived other separation criteria which, as mentioned by the author, are also obtainable from results of Morse, Reid and others on the extension of the Sturmian theory to general self-adjoint systems. In particular, the separation theorem of the author is shown to imply the general separation theorem of the classical Sturm-Liouville theory. The paper concludes with an application of the earlier results to the problem of approximation of eigenvalues, with particular attention to a problem occurring in the study of the transverse vibration of a beam.

W. T. Reid (Evanston, Ill.).

Šnol', È. Behavior of eigenfunctions and the spectrum of Sturm-Liouville operators. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 113–132 (1954). (Russian)

Let $L(y)=-y''(x)+q(x)y(x)$. The author considers $Ly=\lambda y$ on $a\leqq x\leqq b$, where a or b can be unbounded. The author gives an account of work of his own and includes results of Gelfand, Levitan, Sears, Putnam, Hartman, Wintner, Glazman, Molchanov, Naimark, and Krein on the nature of the spectrum and eigenfunctions of the boundary-value problems of $Ly=\lambda y$. *N. Levinson.*

Titchmarsh, E. C. On the asymptotic distribution of eigenvalues. Quart. J. Math., Oxford Ser. (2) **5**, 228–240 (1954).

The author considers the differential equation

$$d^2\psi/dx^2+\{\lambda-q(x)\}\psi=0,$$

with $q(x)$ a monotone increasing function on $[0,\infty)$ such that $q(x)\to\infty$ as $x\to\infty$, so that a singular boundary problem on $[0,\infty)$ determined by a real linear homogeneous boundary condition at $x=0$ has a discrete spectrum consisting of real eigenvalues $\lambda_0<\lambda_1<\cdots$. If $x=X(\lambda)$ is the positive root of the equation $q(x)=\lambda$, and

$$z(\lambda)=\int_0^{X(\lambda)}\{\lambda-q(x)\}^{1/2}dx$$

for λ a real value exceeding the minimum of $q(x)$ on $[0,\infty)$, then under the assumption that $q(x)$ is of class C''', $q(x)$ and $q'(x)$ are non-decreasing, and q'/q, q''/q', q'''/q'' are individually $O(1/x)$ as $x\to\infty$, it is shown that: (a) $z(\lambda_n)=(n+3/4)\pi+O(1/n)$ for the boundary problem involving $\psi(0)=0$; (b) $z(\lambda_n)=(n+1/4)\pi+O(1/n)$ for a boundary problem involving $\psi(0)\cos\alpha+\psi'(0)\sin\alpha=0$ with $\sin\alpha\neq0$. Correspondingly, for the singular boundary problem with interval $(-\infty,\infty)$, and $q(x)$ satisfying similar conditions at both ends of the interval, it is established that for $n=0,1,\cdots$ the eigenvalue $\lambda=\lambda_n$ satisfies

$$\int_{X'}^{X}\{\lambda-q(x)\}^{1/2}dx=(n+1/2)\pi+O(1),$$

where $X=X(\lambda)$ and $X'=X'(\lambda)$ are respectively the positive and negative roots of $q(x)=\lambda$. *W. T. Reid.*

Biglov, Z. I. On a differential operator generated by a system of differential expressions of second order. Dokl. Akad. Nauk SSSR (N.S.) **99**, 495–497 (1954). (Russian)

The author considers $l(y)=-y''+P(x)y$, where $P(x)$ is a real symmetric matrix and $y(x)$ is a vector with n components, on the interval $0\leqq x\leqq\infty$. Self-adjoint operators associated with l are set up and expansion theorems are obtained. In case $P(x)$ is summable on $(0,\infty)$ the defect is associated entirely with the conditions at $x=0$ and is (n,n). Many other results are stated. *N. Levinson.*

Zadiraka, K. V. Construction of two-sided approximations for the eigenvalues of a homogeneous boundary problem of the 4th order. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 243–249 (1954). (Ukrainian. Russian summary)

The equation considered is $d^4y/dx^4+q(x)y=\lambda r(x)y$, first with initial values given; second with boundary conditions of the form

$$\alpha y(0)+\beta y'''(0)=0,\quad \gamma y'(0)+\delta y''(0)=0,$$
$$\alpha_1 y(L)+\beta_1 y'''(L)=0,\quad \gamma_1 y'(L)+\delta_1 y''(L)=0.$$

For each a sequence of upper and one of lower functions is obtained, converging with derivatives to the solution and its derivative. The expressions are rather involved. An illustrative example with

$$q=0,\quad r=1+x,\quad y(0)=y''(0)=y(1)=y''(1)=0$$

is worked out. *A. S. Householder.*

Zadiraka, K. V. A method of computing upper and lower approximations for eigenvalues of a boundary problem of the 2d order. Ukrain. Mat. Ž. **6**, 190–201 (1954). (Russian)

The author begins by exhibiting functions $\varphi(y)$ such that the substitution $z'=z\varphi(y)$ replaces a Sturm-Liouville problem by that of solving an equation of the form $y'=f(x,y,\lambda)$ with boundary conditions $y(a)=y_a$, $y(b)=y_b+k\pi$ with k a positive integer. The method to be described begins with the development of a sequence of upper and one of lower functions converging, for fixed λ, to a solution of the initial-value problem. If $m(\lambda)<\partial f/\partial y<M(y)$, the first sequence is obtained by writing $y'-my=f-my$ and defining the sequence by

$$y_{n+1}(x,\lambda)=y_a+\exp(mx)\int_a^x\exp(-m\xi)[f(\xi,y_n,\lambda)-my_n]d\xi.$$

If y_1 is an upper function then, for every n, $y_n\geqq y_{n+1}\geqq y$ and the sequence converges. Analogously a sequence of lower functions can be formed. Now let $\lambda_n^{(k)}$ be the greatest solution of $y_n(b,\lambda)=y_b+k\pi$. The sequence $\lambda_n^{(k)}$ for fixed k approaches monotonically the limit $\lambda^{(k)}$ which is the kth eigenvalue of the system. *A. S. Householder.*

Feller, William. On second order differential operators. Ann. of Math. (2) **61**, 90–105 (1955).

The operator $Af=af''+bf'+cf$ with $a(x)\geqq0$ has two characteristic properties: (i) local property, that is, the

value $(Af)(x_0)$ is determined by the values of $f(x)$ in any small neighbourhood of x_0; (ii) whenever f has a local minimum at x_0 and $f(x_0)=0$, then $(Af)(x_0)\geqq 0$. The author derives the most general operator (whose domain and range consists of continuous functions) with these two characteristic properties. To eliminate the cumbersome term corresponding to cf, he introduces the "reduced operator" Ω by $\Omega f=A^*f-ef$, $A^*f=\varphi^{-1}A(f\varphi)$, $e=\varphi^{-1}A\varphi$, where $\varphi>0$ is in the domain of A; thus $Af=\varphi\Omega(f\varphi^{-1})+cf$. An interval I is called regular if, and only if, $\Omega z=0$ has a strictly increasing solution z and Ω does not vanish at any point of I. It is shown that, if we take this solution z as the independent variable x, each $f(x)$ in the domain of Ω is the difference of two convex functions. By virtue of this fact it is proved that there exist two strictly increasing functions u and v, of which u is a continuous function, such that $\Omega f=D_vD_uf$, where D_w denotes differentiation with respect to the strictly monotone function w. It is also shown that the functions u and v are intrinsically connected with the operator A and that the formal adjoint of D_vD_u is D_uD_v, though in general D_v and D_u are not commutative with each other. *K. Yosida.*

Partial Differential Equations

Glenn, Oliver E. Invariants when the transformation is infinitesimal, and their relevance in bio-mathematics and in the theory of terrestrial magnetism. Ann. Scuola Norm. Super. Pisa (3) **8**, 1–42 (1954).

This paper consists of three parts. The first part expands the mathematical theory of invariants under infinitesimal transformations, i.e. the first step is to find invariants $H(\varphi, r, \theta, d\varphi, dr, d\theta)$ such as $H'=MH$ (M: const.) under a given infinitesimal transformation U in a three-dimensional space referred to a polar coordinate system

$$(\varphi, r, \theta):\ \varphi'=\varphi+uO(\varphi, r, \theta),\quad r'=r+vP(\varphi, r, \theta),\quad \theta'=\theta+wQ(\varphi, r, \theta)$$

in which $|u|$, $|v|$, $|w|$ are equal to 0. Evidently H must satisfy the equation $\Omega H=0$, where

$$\Omega=uO\frac{\partial}{\partial\varphi}+vP\frac{\partial}{\partial r}+wQ\frac{\partial}{\partial\theta}+u(dO)\frac{\partial}{\partial(d\varphi)}+v(dP)\frac{\partial}{\partial(dr)}+w(dQ)\frac{\partial}{\partial(d\theta)}+(1-M),$$

and we get three particular integrals:

$(d\varphi)/O=\gamma_1$, $(dr)/P=\gamma_2$, $(d\theta)/Q=\gamma_3$ (γ_i: arbitrary const.)

of the auxiliary system of Lagrange, out of six requisite for a complete integral of $\Omega H=0$. The author emphasizes especially the particularizations: if $O=o(\varphi)$, $P=p(r)$, $Q=q(\theta)$, where $o(\varphi)$, $p(r)$, $q(\theta)$ are polynomials of their arguments respectively, then such an infinitesimal transformation U_1 generates a group $\{U_1\}$ and three additional particular integrals of $\Omega H=0$ are obtained. If H is a quantic in the differentials $d\varphi$, dr, $d\theta$, it is of the form:

$$H=\Psi(e^{I(\varphi)}, e^{J(r)}, e^{K(\theta)}, (d\varphi)/o(\varphi), (dr)/p(r), (d\theta)/q(\theta))\quad(\Psi\text{: arbitrary function}),$$

where

$$I(\varphi)=\int\frac{d\varphi}{io(\varphi)},\quad J(r)=\int\frac{dr}{jp(r)},\quad K(\theta)=\int\frac{\theta d}{kq(\theta)}\quad(i, j, k\text{ const.}).$$

The three differential arguments of the type $(d\varphi)/o(\varphi)$ are absolute invariants of $\{U_1\}$ and the other three functional arguments of the type $e^{I(\varphi)}$ relative invariants. Special attention is paid to the case that H is linearly factorable and its factors are

$$I_l=A_l\frac{d\varphi}{o(\varphi)}+B_l\frac{dr}{p(r)}+C_l\frac{d\theta}{q(\theta)}\quad(l=1, 2, \cdots, h)$$

with constants A_l, B_l, C_l. Then the equation $H=0$ represents h surfaces which are respectively invariant under h subgroups of $\{U_1\}$ (these subgroups may not all be mutually exclusive). The second step is to discuss invariant curves lying on these h surfaces and to generalize the above-stated theory to the case that H is a quantic in dx, dy, dz, $d\pi$, $d\lambda$, where π, λ are the variables of direction of a surface element through the point (x, y, z). The last step is the inverse invariant theory, i.e. when the invariant is given, it is required to determine the transformation. Thereby the most general case (n variables $x_1, \cdots, x_n$; $n-1$ partial slopes $\lambda_1, \cdots, \lambda_{n-1}$ and $M=M(x, \lambda, u)$) is taken into consideration, and the total (necessary and sufficient) condition in order that $H_m=0$ should define a generalized connection of surface elements and $H_m'=MH_m$ a species of generalized contact transformation, is obtained, provided that H_m is a quantic in dx's and $d\lambda$'s. As the author indicates, this concept of generalized connection and contact transformation is in a close relation with two recent papers [A. Kawaguchi, Tensor (N.S.) **2**, 123–142 (1952); MR **14**, 585; Hlavatý, Rend. Circ. Mat. Palermo (2) **1**, 403–438 (1953); MR **15**, 349].

In the second part the theory is applied to bio-mathematics, i.e. variations of biological characters, their stabilization and Lamarck's first law, and the last part is an application to the earth's magnetic field of force, considered as directional-central force, making use of the equations of the orbit of the cathode particle derived from its property of stability and from the acceleration. Also, the magnetic force at the pole of the earth and the parameters of the force are discussed. Finally, there are some statements on the formulary and the phenomena. *A. Kawaguchi* (Sapporo).

Hornich, Hans. Überall unlösbare lineare partielle Differentialgleichungen. Monatsh. Math. **59**, 34–42 (1955).

Although the ordinary differential equation (1) $y'=\phi(x, y)$ has a solution through each point of a region G in which ϕ is continuous, the corresponding result is not true for the linear partial differential equation (2) $u_x+\phi(x, y)u_y=f(x, y)$, when ϕ and f are assumed merely continuous in G. Examples of such equations which are not solvable in the neighborhood of a point through which pass more than one solution of the equation (1) have been given, but the point of this paper is to give an example which has no solution throughout the region. Specifically, the following theorem is proved: In every bounded region G of the (x, y)-plane, a continuous function $\phi(x, y)$ and a linear function $f(x, y)$ can be constructed for which equation (2) is unsolvable everywhere in G; that is, in no subregion G' of G does there exist a differentiable function $u(x, y)$ satisfying (2). The construction of the function ϕ is ingenious, depending on its definition at a sequence of points Q_n which are everywhere dense in a circular disc B and have the further property that there are two solutions of (1) through each Q_n. The author also shows that for such a given ϕ, the set of linear functions f for which (2) is solvable form only a two-dimensional linear subspace of the three-dimensional space of linear functions

f, and hence (2) is unsolvable for "almost all" linear functions f. *D. L. Bernstein* (Rochester, N. Y.).

Bureau, Florent J. Divergent integrals and partial differential equations. Comm. Pure Appl. Math. **8**, 143–202 (1955).

Dans une première partie l'A. rappelle les définitions des parties finies (pf) et des parties logarithmiques (pl) d'intégrales divergentes, d'abord pour une variable, puis pour p variables, p quelconque. Il étudie la dérivation sous les signes pf et pl, et donne des exemples. L'A. donne ensuite de nombreuses applications de ces notions:

1) Problème de Cauchy pour l'équation des ondes. On part de la remarque que la fonction $(t^2-x^2)^{-(p-1)/2}$, $x^2=x_1^2+\cdots+x_p^2$, est solution usuelle (en dehors des singularités) de l'équation des ondes. Si l'on cherche u, solution dans $t\geqq 0$ de $u_{tt}-\Delta u=0$, avec $u(x;0)=0$, $\partial u(x;0)/\partial t=g(x)$ (le cas général se ramène à ce cas), on en déduit, grace aux notions de la première partie que: si p est pair, la solution est

$$u(x;t)=A_p\,\mathrm{pf}\int_{|x-y|\leq t} g(y)(t^2-r^2)^{-(p-1)/2}dy,\quad r^2=|x-y|^2,$$

où $A_p=(-1)^{(p-2)/2}\frac{1}{2}\pi^{-(p+1)/2}\Gamma(\frac{1}{2}(p-1))$; si p est impair, la solution est

$$u(x;t)=B_p\,\mathrm{pl}\int_{|x-y|\leq t} g(y)(t^2-r^2)^{-(p-1)/2}dy,$$

où $B_p=(-1)^{(p-1)/2}\frac{1}{2}\pi^{-(p-1)/2}\Gamma(\frac{1}{2}(p-1))$. Raccord avec les solutions usuelles.

2) Problème de Cauchy pour l'équation $u_{tt}-\Delta u-u=0$. Solution par une méthode analogue à celle du 1).

3) Problème de Cauchy pour l'équation

(a) $\quad u_{tt}+kt^{-1}u_t-\Delta u=0$ (Euler-Poisson-Darboux),

k complexe. On part de la remarque que la fonction $(t^2-x^2)^{-(k+p-1)/2}$ est solution usuelle (en dehors des singularités). Conséquence: si $-\mathrm{Re}\,k$ est assez grand,

$$\int_{|x-z|\leq t} u_0(y)(t^2-r^2)^{-(k+p-2)/2}dy$$

est solution de (a), avec $u(x;0)=u_0(x)$, $u_t(x;0)=0$.

Si $u^{(k)}$ est solution de (a), on a la relation suivante, due à Darboux [cf. Weinstein, Bull. Amer. Math. Soc. **59**, 454 (1953)]: $u^{(k)}=t^{1-k}u^{(1-k)}$; on en déduit que

$$U_k^*(x;t)=t^{1-k}\int_{|x-y|\leq t} u_0(y)(t^2-r^2)^{(k-p-1)/2}dy$$

est solution de (a). On effectue ensuite le prolongement analytique en k de $U_k^*(x;t)$:

1. $\frac{1}{2}(k-p-1)$ non entier négatif. On prolonge à l'aide des pf; $k=-2q-1$, $q\geqq 0$ entier, est exceptionnel, relativement aux conditions initiales.

2. $\frac{1}{2}(k-p-1)$ entier négatif. On prolonge à l'aide des pl. Valeurs exceptionnelles comme en 1.

3. $k=-2q-1$, $q\geqq 0$ entier (valeurs exceptionnelles) [cf. aussi Weinstein, loc. cit.; E. K. Blum, ibid. **59**, 345 (1953); Proc. Amer. Math. Soc. **5**, 511–520 (1954); MR **16**, 137; Diaz et Weinberger, ibid. **4**, 703–715 (1953); MR **15**, 321]. Posant $t^2=s$, on utilise essentiellement le fait que $u^{(-2q-1)}=2^{q+1}1_5{}^{q+1}\partial^{q+1}u^{(1)}/\partial s^{q+1}$ (rajouter "(23, 1)", p. 162, ligne 3 à partir du bas), pour effectuer le prolongement.

4) Généralisation du 3) en remplaçant Δ par un opérateur différentiel L, elliptique du 2ème ordre sur un espace de Riemann orientable, analytique, à p dimensions.

5) Généralisation des considérations précédentes à des opérateurs $L(\partial/\partial x_1,\cdots,\partial/\partial x_p;\partial/\partial t)$ totalement hyperboliques. Exemple:

$$L=D_1D_2,\quad D=\frac{\partial^2}{\partial t^2}-\Delta\ (\text{dans } R^3),$$

$$D=\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)-\frac{\partial^2}{\partial x_3^2}.$$

[Cf. Bureau, Acad. Roy. Belg. Bull. Cl. Sci. (5) **33**, 379–402 (1947); MR **9**, 356.]

6) Pour les équations d'évolution hyperboliques [Petrowsky, Bull. Math. Univ. Moscou **1**, no. 7 (1938); Schwartz, Ann. Inst. Fourier, Grenoble **2**, 19–49 (1951); MR **13**, 242] la résolvante est une fonction de t à valeurs dans l'espace des distributions en x à support compact; elle peut donc être calculée à l'aide de la transformation de Fourier-Schwartz [Schwartz, Théorie des distributions, t. **1**, 2, Hermann, Paris, 1950, 1951; MR **12**, 31, 833]. L'A. retrouve ce résultat sur des exemples, en utilisant les notions de pf, pl, et une formule de Parseval généralisée (qui revient à considérer la transformation de Fourier-Schwartz).

7) Construction de solutions élémentaires d'équations elliptiques. Comme application, dans le cas du laplacien, l'A. donne des formules de pf, généralisant des formules données dans l'ouvrage cité de Schwartz.

8) L'A. considère les systèmes différentiels du type suivant sur R^p; à un vecteur $u=(u_1,\cdots,u_p)$, il fait correspondre le vecteur $v=(v_1,\cdots,v_p)$, avec

$$v_j=\sum_{i,\alpha} a_\alpha{}^{ij}D^\alpha u_i+\sum_{i,k,\beta} b_\beta{}^{i,j,k}D^\beta u_i,$$

où les sommes sont étendues à: $i=1,\cdots,s$; $\alpha=(\alpha_1,\cdots,\alpha_p)$, $|\alpha|=\alpha_1+\cdots+\alpha_p=n$; $\beta=(\beta_1,\cdots,\beta_p)$, $|\beta|=k<n$, $k=1,\cdots,n-1$; les a et b sont des constantes. L'A. a montré [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **15**, no. 3 (1936)] qu'on peut associer à ce système un opérateur différentiel B, tel que la connaissance d'une solution élémentaire de B entraine la connaissance d'une matrice solution élémentaire du système. L'A. donne deux applications de ce résultat: a) à un système tel que l'opérateur B associé soit celui considéré en 5); b) à un système d'équation de C. W. Oseen [Ark. Mat. Astr. Fys. **3**, nos. 20, 24 (1907)].

J. L. Lions (Nancy).

Martin, A. I. The extension of two-dimensional eigenfunction expansions to higher dimensions. Proc. London Math. Soc. (3) **4**, 437–455 (1954).

This paper is concerned with expansion theorems for the boundary problem involving the differential equation

$$\partial^2\varphi/\partial x_1^2+\cdots+\partial^2\varphi/\partial x_N^2+\{\lambda-q(x_1,\cdots,x_N)\}\varphi=0,$$

with the region of consideration the whole $(x_1,\cdots,x_N)$-space. The theory of Titchmarsh [Proc. London Math. Soc. (3) **1**, 1–27 (1951); MR **13**, 241] for $N=2$ is modified to obtain corresponding results for the higher-dimensional problem. Indeed, after preliminary results on iterated kernels, which are employed to overcome the difficulty introduced by the fact that for $N\geqq 4$ the Green's function is no longer of class L^2 over its singularity, explicit treatment is limited to detailing suggested modifications to be made in the respective sections of Titchmarsh's paper. In the expansion theorem for $N\geqq 4$ it is found that greater restric-

tions are needed on the function being expanded than are necessary on the analogous functions for $N=2$ or $N=3$.

W. T. Reid (Evanston, Ill.).

Titchmarsh, E. C. Eigenfunction expansions associated with partial differential equations. V. Proc. London Math. Soc. (3) **5**, 1–21 (1955).

[For parts I–IV see MR **13**, 241; **15**, 229; **16**, 43.] In this continuation of earlier papers on expansion theorems for boundary problems involving the differential equation

$$(*) \qquad \nabla^2\psi+\{\lambda-q(x,y)\}\psi=0,$$

and for similar equations in more than two dimensions, the author is concerned with the separability of (*) under a transformation $x=x(u,v)$, $y=y(u,v)$ implying that all eigenfunctions of the boundary problem are of product form $\psi(u,v)=\chi(u)\omega(v)$. This result is established for (*) in polar coordinates (r,θ) and q a function of r alone, with region of consideration the circular region $r\leqq R$ or the entire plane, as well as for the case $q(x,y)=k(x)+l(y)$. Treatment of the case in which the spectrum is only partially discrete involves relationships between various Green's functions. A similar result is obtained for the corresponding three-dimensional equation in spherical polar coordinates with q a function of radius vector only. The latter portion of this paper is devoted to the derivation of similar formulae for the three-dimensional equation that arises in Schrödinger's theory of the hydrogen atom in an electric field. *W. T. Reid.*

Martin, A. I. On L^2-solutions of the wave equation. Quart. J. Math., Oxford Ser. (2) **5**, 212–227 (1954).

For the differential equation

$$(*) \qquad \nabla^2\varphi+\{\lambda-q(x,y)\}\varphi=0,$$

with $q(x,y)$ a real-valued continuous function and region of consideration the entire (x,y)-plane, the author is concerned with the derivation of results corresponding to Weyl's invariability theorem for second-order linear ordinary differential equations. The presented treatment is based on results of Titchmarsh [Proc. London Math. Soc. (3) **1**, 1–27 (1951); MR **13**, 241] concerning the existence and properties of a Green's function $G(x,y,\xi,\eta,\lambda)$ for (*). It is shown that if $G(x,y,\xi,\eta,\lambda)$ is unique for one non-real value λ, then it is unique for all non-real λ. Moreover, in case this Green's function is unique for non-real λ, then: (i) equation (*) has an L^2-solution if and only if λ lies in the point spectrum, i.e., $H(x,y,\xi,\eta,\lambda+0)-H(x,y,\xi,\eta,\lambda-0)\neq 0$, where

$$\pi H(x,y,\xi,\eta,u)=\lim_{v\to 0}\int_0^u \operatorname{Im} G(x,y,\xi,\eta,u'+iv)du';$$

(ii) if λ is real, then $\nabla^2\varphi+\{\lambda-q(x,y)\}\varphi=f$ has a (unique) L^2-solution for arbitrary $f(x,y)$ of L^2 if and only if λ is a non-spectral value, i.e., H is constant in some neighborhood of this value of λ. *W. T. Reid* (Evanston, Ill.).

Glazman, I. M. On an application of the method of decomposition to multidimensional singular boundary problems. Mat. Sb. N.S. **35**(**77**), 231–246 (1954). (Russian)

The author extends to the multidimensional case the method of decomposition previously considered by the author [Dokl. Akad. Nauk SSSR (N.S.) **80**, 153–156 (1951); **87**, 5–8 (1952); MR **13**, 654; **14**, 1088] for the study of the nature of the spectrum for one-dimensional singular boundary-value problems.

The operator $l[u]=-\Delta u+q(P)u$ considered in a domain Ω is called singular if Ω is unbounded or if $q(P)$ is not continuous in $\bar{\Omega}$. The case where Ω is bounded by a closed surface σ is considered and the unbounded Ω is denoted by $\Omega(\sigma,\infty)$. Self-adjoint operators $\tilde{L}$ associated with l on $\Omega(\sigma,\infty)$ are considered. The effect on the continuous spectrum is considered for a change in σ. A change in $l=l_1$ to $l=l_2$ by modifying $q=q_1(P)$ to $q_2(P)$, where $q_2(P)-q_1(P)\to 0$ as $P\to\infty$ leaves the continuous spectrum of $\tilde{L}_1$ identical with that of $\tilde{L}_2$ if the domain of $\tilde{L}_2$ is that of $\tilde{L}_1$. Further results are given.

N. Levinson (Cambridge, Mass.).

Lopatinskiĭ, Ya. B. The behavior of solutions of a linear elliptic system in the neighborhood of an isolated singular point. Dokl. Akad. Nauk SSSR (N.S.) **79**, 727–730 (1951). (Russian)

Soit le système différentiel

$$A(x,\partial/\partial x)=\|A_{kl}(x,\partial/\partial x)\| \quad (k,l=1,\cdots,p),$$
$$A_{kl}(x,\partial/\partial x)=\sum_{|j|\leq s} f^j{}_{kl}(x)D_x{}^j,$$

$f^j{}_{kl}$ étant des fonctions $(j+t)$ fois continûment différentiables dans un domaine D de R^n $(x\in D)$, $t\geqq 0$ fixé. On suppose le système elliptique au sens: dét. $(\sum_{|j|=s} f^j{}_{kl}(x)\alpha^j)\neq 0$, pour tout $x\in D$, et pour tout $\alpha=(\alpha_1,\cdots,\alpha_n)\neq 0$.

Soit V borné contenu dans D. Il existe une matrice élémentaire $\omega(x,y)$, $x,y\in V$, $x\neq y$, étudiée précédemment par l'A. [mêmes Dokl. **71**, 433–436 (1950); MR **11**, 725] (cf. aussi les travaux de F. John, notamment). Soit $\psi(x,y)$ un vecteur dans $V\times V$ en dehors de la diagonale, avec: a) $\psi(x,y)$ est s fois continûment différentiable en x, $x\neq y$; b) $|x-y|^{t-s+|k|+n}D_x{}^k\psi(x,y)$ est bornée dans $V\times V$, $|k|\leqq s-1$ $(n\geqq 3)$; c) $A(x,\partial/\partial x)\psi(x,y)=0$. [Condition b) sous toute réserve quant à l'exposant; il y a une erreur typographique.]

L'A. donne une décomposition simple de $\psi(x,y)$, en utilisant $\omega(x,y)$. Il en déduit notamment: si $|x-y|^{n-1}D_x{}^k\psi(x,y)$ lorsque $x\to y$, pour tout k avec $|k|\leqq s-1$, alors $\psi(x,y)$ est t fois continûment différentiable en x dans V.

J. L. Lions (Nancy).

***Morrey, C. B., Jr. Second-order elliptic systems of differential equations.** Contributions to the theory of partial differential equations, pp. 101–159. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Inspired by the calculus of variations, the author considers elliptic systems

$$(A) \qquad D_\alpha(a^{\alpha\beta}D_\beta u+b^\alpha u+e^\alpha)=c^\alpha D_\alpha u+du+f \qquad (D_\alpha=\partial/\partial x^\alpha,\ \alpha=1,\cdots,\nu)$$

in a suitably integrated form. Here u, e and f are functions of $x=(x_1,\cdots,x_n)$ with values in an N-dimensional euclidean space; T and $a^{\alpha\beta}$, b^α and d are functions whose values are linear transformations of T into itself. That (A) is elliptic means that $a^{\alpha\beta}\lambda_\alpha\lambda_\beta$ is non-singular at every point for all $\lambda=(\lambda_1,\cdots)\neq 0$. The main part of the paper gives estimates, too complicated to be stated here, for a solution or certain square mean values of it in terms of its Dirichlet integral and square integral under weak assumptions about the coefficients. In particular, it is shown that if $a^{\alpha\beta}$, b^α and e^α are of class $C_\mu{}^1$ and c^α, d and f of class $C_\mu{}^0$, then u is of class $C_\mu{}^2$ almost everywhere. Here $C_\mu{}^k$ means the set of functions in C^k whose derivatives of order k satisfy a Hölder condition of order μ. Part of the results were proved for $N=1$ by Schauder [Enseignement Math. **35**, 126–139 (1936)], E. Hopf [Math. Z. **30**, 404–413 (1929)] and the author [Trans. Amer. Math. Soc. **43**, 126–166 (1938); Univ. California Publ. Math. (N.S.) **1**, 1–130 (1943); MR

6, 180]. Applications to non-linear equations are also given. One section deals with Dirichlet's problem for (A) for a bounded region G. This problem is shown to be regularly solvable in the Fredholm sense if $a^{\alpha\beta}\lambda_\alpha\lambda_\beta$ is uniformly positive definite for all λ, $a^{\alpha\beta}$ is uniformly continuous in G, b^α, c^α and d are bounded and measurable and e^α and f square integrable. As in recent work on elliptic equations of higher order [see, e.g., Gårding, Math. Scand. 1, 55–72 (1953); MR 16, 366], the proof is based on a positivity property of the Dirichlet integral given by $a^{\alpha\beta}$, which is proved by combining a Fourier transform argument [L. Van Hove, Nederl. Akad. Wetensch., Proc. 50, 18–23 (1947); MR 8, 522] and a partition of unity. *L. Gårding* (Lund).

Hellwig, Günter. Über partielle Differentialgleichungen zweiter Ordnung von gemischtem Typus. Math. Z. 61, 26–46 (1954).

Suppose the equation

$$Lu \equiv (AU_x)_x - (CU_y)_y - Du = F$$

is hyperbolic for $x>0$, elliptic for $x<0$ and parabolic on the y-axis. The coefficients A, C, D, F are assumed to have certain growth properties in a neighborhood of the y-axis, $x>0$. Under various assumptions on the relative growths of these coefficients the initial-value problem is solved in the hyperbolic portion of the plane with initial values on the parabolic line. Estimates for the solution are obtained in terms of initial data. The method employed reduces the problem to a singular integral equation of Volterra type and this is solved by iteration. This extends previous results of Haack and the author [Arch. Math. 5, 60–76 (1954); MR 16, 255]. If A and C are product functions and $D \equiv F \equiv 0$, the Dirichlet problem is solved for a rectangle in the elliptic portion of the plane with one of the sides situated on the parabolic line. *M. H. Protter* (Berkeley, Calif.).

Hove

***Lax, P. D. The initial value problem for nonlinear hyperbolic equations in two independent variables.** Contributions to the theory of partial differential equations, pp. 211–229. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954.

Exposé de résultats récents obtenus (un grand nombre d'entre eux par l'auteur) dans la résolution du problème de Cauchy pour les systèmes d'équations hyperboliques à deux variables:

$$(1)\quad u_t + Au_x + B = 0, \quad u(x,0) = \Phi(x) \quad (-\infty < x < +\infty)$$

(u vecteur inconnu; A, B matrices dépendant de x, t, u; Φ vecteurs donné). Les parties 1 et 2 concernent les solutions locales, et 3 des solutions faibles globales.

1. L'auteur rappelle la solution du problème (1) à l'aide de majorations dues à Haar [cf. Schauder, Comment. Math. Helv. 9, 263–283 (1937)] pour A, B, Φ suffisamment différentiables. L'existence de la solution pour A, B, Φ à dérivées premières continues a été montre par Douglis [Comm. Pure Appl. Math. 5, 119–154 (1952); MR 14, 655] puis Hartman et Wintner [Amer. J. Math. 74, 834–864 (1952); MR 14, 475]. L'auteur montre l'existence d'une solution généralisée par fermeture dans la norme C_0 (convergence uniforme) pour Φ seulement continue lipshitzienne. Cette solution est continue lipshitzienne tant qu'elle existe. Les discontinuités de Φ' se propagent le long des caractéristiques [Lax, Comm. Pure Appl. Math. 6, 231–258 (1953); MR 15, 36].

2. Emet l'hypothèse que toute solution limite de solution usuelle est lipshitzienne continue à moins que le système n'appartienne à une classe exceptionnelle (dont font partie des équations de la mécanique des fluides compressibles): la pente de la $n$ème caractéristique peut être exprimée comme fonction de $x, t, v^1, \cdots, v^{n-1}$ (nouvelles inconnues telles que leurs dérivées seules interviennent dans les $n-1$ premières équations). L'auteur énonce pour ces systèmes trois théorèmes qui lui permettent de construire par fermeture des solutions avec des vecteurs initiaux dont la $n$ème composant est seulement à variation bornée.

3. Le vecteur fonction u est solution faible de

$$u_t + F_x(x,t,u) + B = 0$$

si le premier membre est nul comme fonctionnelle sur les vecteurs à dérivées premières continues, nuls en dehors d'ensembles bornés [cf. Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR 12, 31, 833], c'est à dire si

$$-\int w(x,0)\Phi dx + \int\int \{-w_t u - w_x F + wB\}\,dx\,dt = 0.$$

L'auteur conjecture qu'il existe une seule solution faible $u = R(\phi)$ dépendant continuement de Φ dans une topologie convenable. En particulier l'équation $u_t + (\frac{1}{2}u^2)_x = 0$ a une solution faible pour Φ mesurable et borné à la limite de solution de l'équation $u_t + (\frac{1}{2}u^2) = \lambda u_{xx}$ [E. Hopf, Comm. Pure Appl. Math. 3, 201–230 (1950); MR 13, 846]. L'auteur montre qu'elle dépend continuement de Φ dans la topologie faible. La méthode des différences lui permet d'autre part le calcul de la solution correspondant à des données initiales discontinues. Il semble bien que cette solution soit la même que précédemment. *Y. Fourès-Bruhat* (Marseille).

Fogel, Karl-Gustav. On the P- and D-phase of the Yukawa potential. Acta Acad. Abo. 19, no. 7, 10 pp. (1954).

The wave equation with the Yukawa potential is solved by an expansion in terms of the coupling constant. This is used to calculate the asymptotic P- and D-phases. A comparison with numerical results of others is made.

N. Levinson (Cambridge, Mass.).

Minasyan, R. S. On the steady distribution of temperature in a prismatic body of hollow rectangular cross-section and finite length. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Nauk 6, no. 5–6, 77–86 (1953). (Russian. Armenian summary)

The author considers the steady distribution of temperature in a hollow cylinder of finite length with rectangular cross-section. The boundary conditions on the bounding faces are chosen so that they are symmetric with respect to planes passing through the axis of the cylinder parallel to two adjacent faces. Thus the problem is reduced to the consideration of a temperature distribution in a solid beam of finite length with an L-shaped cross-section which represents one quarter of the original hollow cylinder, provided the additional condjtions of no flow through the planes representing the cuts are imposed. The L-shaped cross-section is divided into three rectangles and a Fourier expansion of the temperature in each is found. Then by matching these expansions across the boundaries separating the three rectangles and imposing the boundary conditions, a generalized Fourier series solution is obtained in the L-shaped beam, and thus because of symmetry in the hollow cylinder made up of four such beams. *C. G. Maple.*

Knighting, E. On the equation of diffusion in the atmosphere. Quart. J. Mech. Appl. Math. **5**, 423–431 (1952).

This boundary-value problem is solved for $v(z, t)$ in terms of Bessel functions, with the aid of the Laplace transformation: $(z^{l-m}v_z)_z = z^l v_t$ $(z>0, t>0)$, $v \to f(t)$ as $z \to 0$, $v \to \phi(z)$ as $t \to 0$, $v \to 0$ as $z \to \infty$. Here $f(t)$ and $\phi(z)$ are prescribed functions, and l and m are constants $(m>0)$. Meteorological applications of the problem, in which l is generally non-negative, are mentioned and references are given. In the case of heat conduction in an atmosphere whose thermal coefficients depend on the altitude z, $v(x, z)$ represents the temperature at that altitude at time t. In other applications, t may denote the horizontal coordinate. Exact and approximate formulas for flux are presented. *R. V. Churchill.*

Difference Equations

Potters, M. L. A matrix method for the solution of a linear second order difference equation in two variables. Math. Centrum Amsterdam. Rekenafdeling **MR 19**, 14 pp. (1955).

The author develops a technique for solving linear equations of the type arising from the difference equations associated with an elliptic partial differential equation. He illustrates the method by solving the Laplace equation in a square, with given boundary values. The square is covered by a rectangular lattice consisting of $M \cdot N$ points. The partial differential equation is replaced by a system of $M \cdot N$ linear difference equations. The author solves this large system of equations by inverting M matrices of order N.

E. Isaacson (New York, N. Y.).

Mignot, Noël. Formes discrètes linéaires et bilinéaires. Opérateurs sur ces formes, formes adjointes. C. R. Acad. Sci. Paris **240**, 837–839 (1955).

In this short note the notion of adjoint linear differential expressions is generalized. That is, two linear difference expressions $L(f)$ and $M(f)$ are said to be adjoint if the bilinear form $fL(g)-gM(f)$ is an "exact difference". Similarly the notion of a self-adjoint difference expression is introduced. It is stated that the necessary and sufficient condition that $M(f)$ be self-adjoint is that it may be written in the form

$$M(f) = \sum_k \Delta^k [a_k(j) \Delta^k f(j-k)],$$

where Δ is the ordinary finite-difference operator.

E. Isaacson (New York, N. Y.).

Filippov, A. F. On stability of difference equations. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1045–1048 (1955). (Russian)

The author considers approximation of differential equations by difference equations. His results can be extended to systems. Essentially the following is shown. If a solution of the differential equation exists and if the difference equation approximates the differential equation and if the solution of the difference equation is stable (in the conventional sense), then the solution of the latter tends in an appropriate sense to the solution of the former, when the net associated with the difference equation becomes increasingly finer. The above refers to boundary problems of sufficiently regular type. *W. J. Trjitzinsky* (Urbana, Ill.).

Bass, G. I. Formulas for solution of Cauchy's problem for some differential-difference equations. Dokl. Akad. Nauk SSSR (N.S.) **100**, 613–616 (1955). (Russian)

On cherche $u(x, t)$, $x \in R$, $t \geq 0$, solution de

$$\partial u/\partial t(x, t) = \alpha(t) \Delta^l u(x, t)/h^l, \tag{1}$$

où $\alpha(t)$ dépend continûment de t, l est un entier positif, $\Delta\varphi(x) = \varphi(x+\frac{1}{2}h) - \varphi(x-\frac{1}{2}h)$, avec la condition initiale

$$u(x, 0) = u_0(x). \tag{2}$$

On suppose que $|u_0(x)| \leq C \exp(\epsilon|x| \log(1+|x|))$. On résout ce problème par la méthode de transformation de Fourier en x; si $v(s, t)$ est la transformée de Fourier en x de $u(x, t)$, alors $v(s, t)$ est solution d'une équation différentielle usuelle (pour tout s fixé) que l'on résout explicitement. Reste ensuite à calculer la transformée de Fourier inverse de $v(s, t)$; l'A. effectue ce calcul sous forme de développement en série. *J. L. Lions* (Nancy).

Kamenskiĭ, G. A. On the asymptotic behavior of solutions of linear differential equations of the second order with retarded argument. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 195–204 (1954). (Russian)

A systematic investigation of the asymptotic behavior of the differential-difference equations

$$y''(x) - M(x)y(x-\Delta(x)) = P(x) \tag{1}$$

and

$$y''(x) + M(x)y(x-\Delta(x)) = P(x) \tag{2}$$

of unstable and periodic types respectively. The functions $M(x)$, $P(x)$, and $\Delta(x)$ are continuous. $M(x) \geq 0$ for large x and $\int_A^\infty M(x)dx = +\infty$; $\Delta(x) \geq 0$ and $\lim_{x\to\infty}(x-\Delta(x)) = +\infty$. $y(x)$ is a solution of one of the equations if it satisfies it for $x \geq A$ and if $y(x-\Delta(x)) = \phi(x-\Delta(x))$ for all $x \geq A$ for which $x-\Delta(x) < A$, ϕ being a given "initial" function. It is known that given a ϕ and the value of $y'(A)$ the solution of either equation is unique. The behavior of the solutions is examined under various assumptions on $M(x)$, $P(x)$, $\phi(x)$, and $y'(A)$. There are some thirteen theorems, of which the following is typical: Suppose $\liminf_{x\to\infty} M(x) > 0$, $|\int_A^x P(t)dt| < C$ for all $x \geq A$ and $\Delta(x)$ satisfies a Lipschitz condition for $x \geq A$. Then every solution of (1) which changes sign for arbitrarily large values of x satisfies either (a) $\lim_{x\to\infty} |y(x)| = \lim_{x\to\infty} |y'(x)| = \infty$ or (b) $\lim_{x\to\infty} |y(x)| = 0$ and $|y'(x)| < C_1$ for $A \leq x < \infty$. *J. M. Danskin.*

Equations in infinitely many unknowns, Integral Equations

Rado, F. Observations au sujet d'un système linéaire infini. Acad. Repub. Pop. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **5**, 285–292 (1953). (Romanian. Russian and French summaries)

The system of equations considered is (1) $\sum_n a_n^k u_n = A_k$, where $k = 0, 1, \cdots$; $0 \leq a_0 < a_1 < \cdots < a_n \to \infty$; and A_k is a bounded sequence. Borel [Ann. Sci. Ecole Norm. Sup. (3) **12**, 9–55 (1895)] asserts that if the entire function $g(z)$ has simple zeros at $z = a_n$, $g(z)/(z-a_n) = \sum_k c_k^{(n)} z^k$, and if $\sum_n a_n^k / |g'(a_n)| < \infty$, then $u_n = \sum_k c_k^{(n)} A_k / g'(a_n)$ is a solution of the system of equations (1). The author shows that if $A_0 = A_1 = 1$, $A_n = 0$ for $n > 1$, $a_n = n\pi$, $g(z) = e^z \sin z$ then the Borel conditions are satisfied, but the expressions for u_n do not satisfy the system of equations. Borel's statement

should have been: if the system $\sum_n a_n^k u_n = A_k$ has a solution u_n satisfying sufficient convergence conditions to guarantee the interchange of order of infinite summations, then the solution has the form given in (2). Essentially this means that the infinite matrix $c_k^{(n)}/g'(a_n)$ is a left reciprocal for a_n^k (i.e. $\sum_k c_k^{(n)} a_m^k / g'(a_n) = \delta_{nm}$), but it is not in general also a right reciprocal. The author points out that if the homogeneous system has one non-trivial solution, then it has an infinite number of linearly independent solutions. He also gives necessary, as well as sufficient conditions for the existence of solutions of the system (1), the latter involving the statement that the system $\sum_k c_k^{(n)} x_k = 0$ have x_k as only solution. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Chang, Shih-Hsun. Integral equations with normal kernels. Sci. Sinica 3, 369–385 (1954).

If the kernel $K(s, t)$ is in L^2, i.e. $\int_a^b \int_a^b |K(s, t)|^2 ds dt < \infty$, and normal i.e.

$$KK^* = \int K(s, u)\overline{K(t, u)}du = \int \overline{K(u, s)}K(u, t)du = K^*K,$$

then the two systems of Schmidt adjoint functions $\{\phi_i\}$ and $\{\psi_i\}$, where

$$\phi_i(s) = \lambda_i \int K(s, t)\psi_i(t)dt \quad \text{and} \quad \psi_i(s) = \lambda_i \int \phi_i(t)\overline{K(t, s)}dt$$

are linearly equivalent via blocks of unitary transformations of finite order. The then existing characteristic constants μ of K are determined by the corresponding λ with $|\mu_i| = \lambda_i$, and the characteristic functions of K are linear combinations of the ϕ. The Schmidt theory relative to the integral equation $f - \lambda Kf = g$ can be adapted to this more general case. These results are more or less known [see Schmeidler, Integralgleichungen mit Anwendungen in Physik und Technik, Akademische Verlagsgesellschaft, Leipzig, 1950, pp. 186–192, 265–268; MR 13, 466.] *T. H. Hildebrandt.*

Arešev, M. S. On a class of linear integral equations. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 5, 3–10 (1949). (Russian)

Under consideration is the equation

$$(1) \quad u(x) = f(\lambda, x, u_1, \cdots, u_n) + \lambda \int_a^b K(x, y)u(s)\,ds,$$

where $u_i = u(c_i)$, c_i are fixed on $[a, b]$, f is continuous in x on $[a, b]$ and continuous in u in $(-\infty, \infty)$, $K(x, t)$ is regular on $S\{a \leqq x, t \leqq b\}$. If λ is a c.v. (characteristic value) of rank m of $K(x, t)$, there exist characteristic functions $\phi_i(x)$ (on the right), $\psi_i(x)$ (on the left), $i = 1, \cdots, m$. The following is proved. If λ is not a c.v., then (1) is equivalent to

$$(2) \quad u(x) = f(\lambda, x, u_1, \cdots, u_n) + \lambda \int_a^b R(x, s)f(\lambda, s, u_1, \cdots, u_n)\,ds,$$

where $R(x, t)$ is the Fredholm resolvent kernel of $K(x, t)$. If λ is a c.v. of rank m and if $\int_a^b f(\lambda, s, u_1, \cdots, u_n)\phi_i(s)\,ds = 0$ $(i = 1, \cdots, m)$, then (1) is equivalent to

$$u(x) = f + \sum_{i=1}^{m} u_{n+i}\varphi_i(x) + \lambda \int_a^b H(x, s)f(\lambda, s, u_1, \cdots, u_n)\,ds,$$

$$H(x, t) = D(x, x_1, \cdots, x_m, t, t_1, \cdots, t_m) / D(x_1, \cdots, x_m, t_1, \cdots, t_m);$$

here $H(x, t)$ is ratio of Fredholm minors of $K(x, t)$, of orders $m+1$, and m, respectively. The originally unknown constants u_i are determined. *W. J. Trjitzinsky.*

Pennisi, L. L. Fredholm integral equations, the reciprocals of whose solutions are also solutions. Amer. Math. Monthly 62, 113–115 (1955).

The author proves the following result. Let C be a bounded measurable set in R^n, and let $K(x, y)$ be a real-valued function in $L_2(C \times C)$. Suppose that the integral equation

$$(1) \quad u(x) = \int_C K(x, y)u(y)dy$$

has the following properties. (a) There is a real L_2 solution $u_0(x)$ of (1) such that $1/u_0(x)$ is in L_2. (b) If $u(x)$ is an L_2 solution of (1), and $1/u(x)$ is in L_2, then $1/u(x)$ satisfies (1). (c) If $u(x)$ is an L_2 solution of (1), then $[b + u(x)]^{-1}$ is in L_2 for some constant b. Then there are a positive integer m and a decomposition

$$C = C_0 \cup C_1 \cup \cdots \cup C_m$$

of C into measurable sets C_i such that

$$m(C_0) = 0, \quad m(C_i \cap C_j) = 0 \quad (i \neq j)$$

and the characteristic functions $\varphi_i(x)$ of the sets C_i $(i \neq 0)$ form a complete set of linearly independent solutions of (1). *F. Smithies* (Cambridge, England).

Mukminov, B. R. On expansion with respect to the eigenfunctions of dissipative kernels. Dokl. Akad. Nauk SSSR (N.S.) 99, 499–502 (1954). (Russian)

The kernel $K(x, y)$ of a linear integral equation is said to be dissipative if the Hermitian kernel

$$\frac{1}{2i}[K(x, y) - \overline{K(y, x)}] = \Im K(x, y)$$

is non-negative definite. Let $K(x, y)$ be a dissipative kernel of class L^2 such that $\int_a^b \Im K(x, x)dx < \infty$ (presumably some further smoothness condition is implicitly assumed). The author gives an elementary proof of the inequality

$$(1) \quad \sum_{n=1}^{\infty} \Im\left(\frac{1}{\lambda_n}\right) \leqq \int_a^b \Im K(x, x)dx,$$

and of the result that the principal functions of K form a complete system in the range of the associated operator if and only if equality holds in (1).

A system of L^2 functions (φ_n) is said to be almost orthonormal if there exist positive constants m and M such that the inequalities

$$m\|f\|^2 \leqq \sum_{n=1}^{N} |c_n|^2 \leqq M\|f\|^2$$

hold for every finite linear combination $f = \sum_{n=1}^{N} c_n \varphi_n$. Sufficient conditions on the characteristic values of a complete dissipative kernel are given for its principal functions to form an almost orthonormal system. Applications to non-self-adjoint differential equations are indicated. *F. Smithies* (Cambridge, England).

Dumitrescu, Eugeniu. On solving a Volterra integral equation by series development. Gaz. Mat. Fiz. Ser. A. 6, 542–545 (1954). (Romanian)

Gahov, F. D., and Čibrikova, L. I. On some types of singular integral equations solvable in closed form. Mat. Sb. N.S. **35(77)**, 395–436 (1954). (Russian)

The integral equation

$$(1)\quad a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{L_0}[(\tau-t)^{-1}+K(t,\tau)]\varphi(\tau)\,d\tau=c(t),$$

in the complex plane, under appropriate conditions and with integration in the sense of Cauchy principal values, as well as some other equations of similar type have been extensively studied. One method is that of regularization, which consists of an application of a suitable operator to the given equation so as to yield a Fredholm equation. The c.e. (characteristic equation) (2) for (1) is obtained by removing the term $K(t,\tau)$; this equation is solved with the aid of the Plemelj (Sohotsky) formulas for limits of Cauchy integrals at points on the path of integration. The authors present a systematic application of the method that one uses in the case of (2). For this purpose one has to study the limits, at points of L_0, of the integral

$$\Phi(z)=\frac{1}{2\pi i}\int_{L_0}[(t-z)^{-1}+K(z,\tau)]\varphi(\tau)\,d\tau.$$

One has to assume that $K(t,z)$ is single-valued analytic in t. In order that the method in question be effective it is necessary that for all lines of discontinuity of $\Phi(z)$ the equation (1) should admit Riemann boundary problems. On this basis the authors formulate and solve in closed form certain types of singular integral equations. One is led to certain Riemann boundary problems, with the novel feature of the requirement that the solution have a specified analytic expression. To achieve the latter purpose the authors are naturally brought to the use of automorphic functions. One seeks automorphic piece-wise analytic functions which on a finite number of lines of discontinuity satisfy boundary conditions of Riemann type. The essential use of automorphic functions and of the corresponding groups of transformations for the purpose of solving various types of singular integral equations constitutes a fresh approach in the theory of such equations. The study presents a substantial development in this field. Space does not allow presentation of all the results. *W. J. Trjitzinsky* (Urbana, Ill.).

Vasilache, Sergiu. Sur une classe d'équations intégrales singulières qui apparaissent dans la théorie des équations intégro-différentielles. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 541–554 (1954). (Romanian. Russian and French summaries)

In previous papers [MR **16**, 372] the solution of the integro-differential equation

$$\sum_{i=0}^{n}H_i(x)\phi^{(i)}(x)=F(x)+\lambda\int_a^x\sum_{r=0}^{p}K_r(x,j)\phi^{(r)}(s)\,ds$$

has been reduced to the solution of an integral equation in $\phi(x)$. If $n=p-1$, this latter equation is of the first kind, which after the usual differentiation process reduces the determination of $\phi(x)$ to the solution of the equation

$$\phi(x)[\lambda K_p(x,x)+H_{p-1}(x)]$$
$$=h(x)+\lambda g(x)+\int_a^x(L_1(x,s)+\lambda L_2(x,s))\phi(s)\,ds,$$

h, g, L_1 and L_2 known. The note is concerned with the values of λ for which for some x in (a,b): $\lambda K_p(x,x)+H_{p-1}(x)=0$. The methods developed by Volterra and Pérès [Théorie générales des fonctionelles, Gauthier-Villars, Paris, 1936, pp. 180ff.] Picard [Ann. Sci. Ecole Norm. Sup. (3) **28**, 459–472 (1911)] and G. C. Evans [Trans. Amer. Math. Soc. **11**, 393–413 (1910)] are applied. The special equation

$$y(x)=f(x)+\lambda\int_a^x K(x,s)y'(s)\,ds$$

is considered in detail. *T. H. Hildebrandt.*

Sobolev, V. V. On the theory of an unsteady radiation field. II. Astr. Ž. **29**, 517–525 (1952). (Russian)

The problem of his earlier paper [Astr. Ž. **29**, 406–417 (1952); MR **16**, 184] is reconsidered by the author and the integro-differential equations involved are solved again, this time by the use of the Laplace operator.

R. G. Langebartel (Urbana, Ill.).

Functional Analysis

Köthe, Gottfried. Theory of locally convex spaces and their applications to analysis. Gaz. Mat. **15**, no. 59, 1–5 (1954). (Portuguese)

Expository paper.

Hörmander, Lars. Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. Ark. Mat. **3**, 181–186 (1955).

Let E be a locally convex real topological linear space and E' its dual. For a convex subset K of E, the support function H of K is given by $H(y)=\sup_{x\in K}\langle x,y\rangle$ for each $y\in E'$, where $\langle x,y\rangle$ denotes the value at x of the linear functional y. It is observed that several of the standard finite-dimensional results on support functions (for example, those concerning linear combinations or convex hulls of families of sets) can be extended without change to this more general situation. For other results, the usual proofs carry through once the proper topology is discovered. Theorem 5. A function on E' to $]-\infty,\infty]$ is the support function of some closed convex subset of E if and only if it is positively homogeneous, subadditive, and lower semicontinuous for the weak topology $\sigma(E',E)$ on E'. Theorems 6–7. The support function of K is strongly (resp., weakly) continuous on E' if and only if K is bounded (resp., bounded and finite-dimensional).

Now let $\mathfrak{K}$ be the system of all bounded closed convex subsets of E, topologized by means of the natural analogue of the Hausdorff distance, with multiplication by non-negative scalars defined as usual and the sum of two sets defined to be the closure of their linear sum. Let $\mathfrak{C}$ be the locally convex space of all strongly continuous positively homogeneous real functions on E', under the topology of uniform convergence on equicontinuous sets, and let τ map each set $K\in\mathfrak{K}$ onto its support function $H\in\mathfrak{C}$. Theorem 8. The map τ is an algebraic, topological, and order-isomorphism of $\mathfrak{K}$ onto a convex cone in $\mathfrak{C}$. When E is a normed linear space, $\mathfrak{K}$ admits a natural norm under which τ is an isometry of $\mathfrak{K}$ into $\mathfrak{C}$, and thus Theorem 8 extends a result of Rådström [Proc. Amer. Math. Soc. **3**, 165–169 (1952); MR **13**, 659]. It is also remarked that $\tau K-\tau K$ is dense in $\mathfrak{C}$ when E is finite-dimensional, but not when E is a non-reflexive normed linear space. *V. L. Klee.*

Klee, V. L., Jr. Separation properties of convex cones. Proc. Amer. Math. Soc. **6**, 313–318 (1955).

The author is concerned with cones with vertex at ϕ in a linear topological space L; the desired conclusion is that if A and B are two such cones there shall be a linear continuous functional f such that $\inf_{x \in A} f(x) \geqq 0 \geqq \sup_{y \in B} f(y)$, with $>$ whenever it is possible. Under assumption that A has an interior and B does not meet it this (Eidelheit's theorem) has long been known in locally convex spaces [see, e.g., Bourbaki, Espaces vectoriels topologiques, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; MR **14**, 880].

In this note $A' = A \cap (-A)$. Under assumption that L is locally convex, that A is locally compact, and that $A \cap B = (\phi)$, it is shown that a separating f exists with $f(x) > 0$ if $x \in A \backslash A'$. If also L is separable normed or B is locally compact, f can be found with $f(y) < 0$ if $y \in B \backslash B'$.

M. M. Day (Urbana, Ill.).

Kasahara, Shouro. A note on f-completeness. Proc. Japan Acad. **30**, 572–575, 848 (1954).

References: spherical completeness and Hahn-Banach theorem for non-Archimedean vector spaces: A. W. Ingleton, Proc. Cambridge Philos. Soc. **48**, 41–45 (1952) [MR **13**, 659]; Hahn-Banach theorem for non-Archimedean vector spaces: T. Ono, J. Math. Soc. Japan **5**, 1–5 (1953); Nagoya Math. J. **6**, 171–176 (1953); I. Fleischer, Indag. Math. **16**, 165–168 (1954) [MR **15**, 717, 964; and further literature cited there]; local flatness and f-completeness will be discussed in a forthcoming paper of the author. Results: (1) Products and quotients of f-complete spaces are f-complete; if the closed subspace W of the locally flat space E is f-complete and if E/W is f-complete, then so is E; spherical completeness implies completeness. (2) Closed subspaces of non-Archimedean normed spaces with extension property have the extension property; continuity of linear mappings between locally flat spaces E and F being equivalent to boundedness in a certain sense (see the author's paper referred to above), let W be an f-complete subspace of E and u_W a linear mapping of W into F; then u_W can be extended to a mapping u_E of E into F with the same bounds as those of u_W. *G. K. Kalisch* (Minneapolis, Minn.).

Tatarkiewicz, Krzysztof. Sur les limites des coefficients des suites des polynômes généralisés. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **6** (1952), 47–54 (1954). (Polish and Russian summaries)

The author generalizes results of the reviewer (referred to as F. Orin) concerning series expansions in Banach spaces [Amer. J. Math. **63**, 87–100 (1941); MR **2**, 222]. Originally, expansion coefficients for an element f were defined as limits of coefficients of finite linear combinations of elements $\{z_n\}$ converging to f, but only if the system $\{z_n\}$ was minimal. In the present paper the author replaces the condition of minimality by a weaker condition, obtaining results analogous to the previous ones, but more general.

O. Frink (University Park, Pa.).

Lax, Peter D. Symmetrizable linear transformations. Comm. Pure Appl. Math. **7**, 633–647 (1954).

Let B be a Banach space, the norm being denoted by $|x|$. Let (x, y) be a positive definite Hermitian form, defined and continuous in $B \times B$, and inducing a norm $\|x\|$, and let H be the completion of B with respect to this norm. Let T be a bounded linear operator in B satisfying $(Tx, y) = (x, Ty)$ $(x, y \in B)$. The author proves the following results. (i) T is bounded with respect to the Hilbert space norm. (ii) The H-spectrum of T is contained in its B-spectrum. (iii) The point spectrum of T in B is contained in the point spectrum of T in H, and the null-spaces of $T - \lambda I$ in the two spaces are then identical. According to the author's definition, a number λ is said to belong to the point spectrum of T if the dimension of the null-space of $T - \lambda I$ is finite and positive, and equal to the dimension of B/J_λ, where J_λ is the range of $T - \lambda I$. From these results the author deduces that if T or any power of it is completely continuous in B, then T is completely continuous in H.

Applications are given to (a) the boundedness of certain special quadratic forms, (b) the boundedness of the semi-group operators associated with a hyperbolic system of partial differential equations, and (c) the pointwise validity of the boundary conditions for the eigenfunctions of an elliptic differential operator. *F. Smithies.*

Yood, Bertram. Difference algebras of linear transformations on a Banach space. Pacific J. Math. **4**, 615–636 (1954).

Let $\mathfrak{E}$ be the set of bounded linear operators on a Banach space $\mathfrak{X}$, and let $\mathfrak{K}$ be the set of completely continuous linear operators on $\mathfrak{X}$. The author discusses the properties of the difference algebra $\mathfrak{E} - \mathfrak{K}$ (or quotient algebra $\mathfrak{E}/\mathfrak{K}$), of the canonical homomorphism π of $\mathfrak{E}$ onto $\mathfrak{E} - \mathfrak{K}$, and of the radical $\mathfrak{P}_1$ of $\mathfrak{E} - \mathfrak{K}$.

The operator T is said to have property A if T has a closed range and finite nullity; $\mathfrak{H}$ is the set of operators T such that T and T^* both have property A. For $T \in \mathfrak{H}$, define $f(T) = \text{nul}\,(T^*) - \text{nul}\,(T)$; then $f(T)$ is continuous on $\mathfrak{H}$. If $\mathfrak{W}$ is any closed two-sided ideal such that $\mathfrak{K} \subset \mathfrak{W} \subset \pi^{-1}(\mathfrak{P}_1)$, then T will have a two-sided inverse (mod $\mathfrak{W}$) if and only if $T \in \mathfrak{H}$. The set $\pi^{-1}(\mathfrak{P}_1)$ can be characterized as the set of operators U such that $T + U$ has property A for every regular T.

The author also gives a criterion for $\mathfrak{E} - \mathfrak{K}$ to be semi-simple, some generalizations of Schauder's theorem that $f(I + K) = 0$ when K is completely continuous, and conditions for the functional f to be non-trivial. Finally, if $\mathfrak{G}$ and $\mathfrak{G}_1$ are the sets of regular elements of $\mathfrak{E}$ and $\mathfrak{E} - \mathfrak{K}$ respectively, the author shows that $\pi(\mathfrak{G})$ is a normal subgroup of $\mathfrak{G}_1$ and that either $\pi(\mathfrak{G}) = \mathfrak{G}_1$ or $\mathfrak{G}_1/\pi(\mathfrak{G})$ is isomorphic to the additive group of integers with the discrete topology.

F. Smithies (Cambridge, England).

Rosenbloom, Paul. Perturbation of linear operators in Banach spaces. Arch. Math. **6**, 89–101 (1955).

The author is concerned with the perturbation of a (not necessarily bounded) linear mapping T_0 of a complex Banach space X into itself by a (not necessarily linear) continuous mapping U in the neighborhood of a simple eigenvalue. While the general framework is more or less known from the work of B. Sz.-Nagy [Acta Sci. Math. Szeged **14**, 125–137 (1951); MR **13**, 849] and others, the interest here is to obtain concrete quantitative results expressed in terms of explicit constructions. Specifically it is assumed that λ_0 is a simple eigenvalue with eigenvector x_0; that if $x_0^* \in X^*$, $x_0^*(x_0) = 1$, is such that $x_0^*(T - \lambda_0)x = 0$, for all $x \in X$, and if $X_1 = \{x \in X : x_0^*(x) = 0\}$, then $T_0 - \lambda_0$ has a bounded inverse on X_1 to X_1. The aim is to solve the equations $T_0 x + Ux = \lambda x$, $x_0^*(x) = 1$, for (x, λ) near (x_0, λ_0) in terms of U. The main tool is the implicit function theorem of T. H. Hildebrandt and L. M. Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927)] and the process of iterations. Application is made to the equation $d^2x/dt^2 + Ux = \lambda x$, where

U is an operator on $C[0, \pi]$. (On page 91, the second x_0 in line 2, and the x_0 in lines 5 and 12 should be x_0^*.)
R. G. Bartle (Urbana, Ill.).

Krasnosel'skiĭ, M. A. Two remarks on the method of successive approximations. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 123–127 (1955). (Russian)

Let T be a bounded closed convex subset of a Banach space E and let A be a completely continuous function on T into itself. Theorem 1. If E is uniformly convex and A also satisfies a Lipschitz condition with constant equal to unity, then for any choice of x_1 in T the sequence $\{x_n\}$ defined by $x_{n+1}=\frac{1}{2}(Ax_n+x_n)$, $n=1, 2, \cdots$, converges to a fixed-point of A. (The existence of the fixed-point follows without the Lipschitz condition from a theorem of J. Schauder [Studia Math. **2**, 171–180 (1930)]; the concern here is with its calculation.) Theorem 2. Let A be as originally, let B satisfy a Lipschitz condition on T with constant $q<1$, and let $Ax+By \in T$ for $x, y \in T$. Then $A+B$ has a fixed-point in T. The possibility of perturbing A and B by adding small terms is briefly considered.
R. G. Bartle (Urbana, Ill.).

Hille, Einar. Sur un théorème de perturbation. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 169–184 (1954).

Let M denote the Banach space of infinite matrices $B=(b_{ij})$ with the norm $\|B\|=\sup_j \sum_k |b_{jk}|<\infty$, and let K be the totality of infinite matrices $A=(a_{ij})$ satisfying $a_{ij}\geqq 0$ $(i\neq j)$, $a_{ii}\leqq 0$ and $\sum_j a_{ij}=0$. Theorem 1 states the following: Let a triangular matrix U and a matrix $V \in M$ both belong to K. If $Z(t)=(z_{ij}(t))$, $t>0$, belongs to M and satisfies i) strong $\lim_{h\to 0} h^{-1}(Z(t+h)-Z(t))=Z(t)U$, ii) $z_{ij}(t)\geqq 0$, iii) strong $\lim_{t\downarrow 0} Z(t)=0$, iv) $\limsup_{t\to\infty} t^{-1}\log\|Z(t)\|=\omega<\infty$, then the equation

$$\text{strong } \lim_{h\to 0} h^{-1}(Q(t+h)-Q(t))=Q(t)(U+V)$$

admits a solution $Q(t)$ satisfying the similar conditions as for $Z(t)$, ω being replaced by $\max(\omega, \|V\|)$. Theorem 2 says that Theorem 1 holds for $V=\beta B-BU$ with $\beta\geqq 0$ if (1) $\lambda\|(\lambda I-U)^{-1}\|\leqq K$ (with $1\leqq K<\infty$) for $\Re(\lambda)>0$, (2) $\sup_j \sum_m b_{jm}u_{mk}<\infty$ $(k=1, 2, \cdots)$, and (3) $\|B\|<(1+K)^{-1}$.
K. Yosida (Tokyo).

***Hille, Einar. An abstract formulation of Cauchy's problem.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 79–89 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Let U be a linear operator whose domain $D(U)$ and range $R(U)$ belongs to the Banach space X. In an abstract Cauchy's problem (ACP) one wishes to find a solution $y(t)=y(t, y_0)$ for $t>0$ of $U(y(t))=y'(t)$ such that $\lim_{t\to 0}\|y(t)-y_0\|=0$. Here $y(t, y_0)$ is called of normal type if $\limsup_{t\to\infty} t^{-1}\log\|y(t)\|=\omega<\infty$; $y(t)=y(t; 0)$ is called a nul-solution if $y(t)\not\equiv 0$. Theorem 1 states that the ACP for any given y_0 has at most one solution of normal type if U is a closed operator whose characteristic values are not dense in any right half-plane. Theorem 2 states that, when U is a closed operator, a necessary and sufficient condition for the existence of a nul-solution of type $\leqq\omega$ is the existence of a solution of $U(x(\lambda))=\lambda x(\lambda)$ which is $\not\equiv 0$, holomorphic and bounded in any half-plane $\Re(\lambda)>\omega+\epsilon$, $\epsilon>0$. Theorems 3 and 4 give a connection between the ACP and the theory of semi-groups of bounded linear operators. As a concrete application of the results above, two special cases are discussed: (i) $U(g)=C(g)=bg''+ag'$ for $X=C[\alpha, \beta]$ and (ii) $U(h)=L(h)=((bh)'-ah)'$ for $X=L_1(\alpha, \beta)$; here $a(x)$ and $b(x)$ are continuous functions in the open interval (α, β), $-\infty\leqq\alpha<0<\beta\leqq\infty$, with $b(x)>0$. If we put

$$W(x)=\exp\left(\int_0^x -a(s)b(s)^{-1}ds\right),\quad W_1(x)=\int_0^x W(s)ds,$$

$$W_2(x)=\int_0^x W_1(s)b(s)^{-1}W(s)^{-1}ds,$$

$$\Omega(x)=\int_0^x W(s)ds\int_0^s b(t)^{-1}W(t)^{-1}dt,$$

then a necessary and sufficient condition in order that the ACP for the operator C (L) has a unique solution of normal type when $y_0 \in D(C)$ $(D(L))$ is given by $\Omega(\alpha)=\infty$, $\Omega(\beta)=\infty$ $(W_2(\alpha)=\infty,\ W_2(\beta)=\infty)$. Finally, some extensions to the case of the "wave equation" $U(y(t))=y''(t)$ (the uniqueness of the solution and the existence of a nul-solution) are also discussed.
K. Yosida (Tokyo).

Phillips, R. S. Semi-groups of operators. Bull. Amer. Math. Soc. **61**, 16–33 (1955).

This is a survey of some of the recent developments in the theory of one-parameter semi-groups of bounded linear operators (b.l.o.). The developments presented here are mainly due to the author and give extensions and refinements of the results due to E. Hille [Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; MR **9**, 594] and the reviewer [J. Math. Soc. Japan **1**, 15–21 (1948); MR **10**, 462]. §1. Introduction. §2. Generation of semi-groups. Theorem 2.1: A necessary and sufficient condition that a closed linear operator U is the infinitesimal generator of a semi-group $\{T(\xi)\}$, $\xi>0$, of b.l. operators (in a Banach space X into X) satisfying the condition strong $\lim_{x\downarrow 0} T(\xi)x=x$ is that the domain $D(U)$ be dense in X and that there exist real constants M and ω such that $\|(\lambda I-U)^{-n}\|\leqq M(\lambda-\omega)^{-n}$ for all $\lambda>\omega$ and $n=1, \cdots$. Theorem 2.2 generalizes the Theorem 2.1 to the class $(1, A)$ of semi-groups $\{T(\xi)\}$ for which (i) $T(\xi)$ is strongly continuous for $\xi>0$, (ii) $\int_0^1\|T(\xi)\|d\xi<\infty$, and (iii) $\lim_{\lambda\to\infty}\lambda\int_0^\infty \exp(-\lambda\xi)T(\xi)xd\xi=x$. §3. Perturbation theory. Theorem 3.1: Let $\{T(\xi)\}=\{T(\xi: U)\}$ be a semi-group of class $(1, A)$ with the infinitesimal generator U. Then $U_1=U+V$ is the infinitesimal generator of a semi-group of class $(1, A)$ if (1) $D(V)=D(U)$, (2) $V(\lambda I-U)^{-1}$ is a b.l.o. for some λ in the resolvent set of U and (3) $\int_0^1\|VT(\xi: U)\|_v d\xi<\infty$ ($\|\ \|_v$ denotes the norm taken relative to the subspace $D(U)$). §4. Semi-group of class (A) is a class between the class $(1, A)$ and Feller's class [Ann. of Math. (2) **58**, 166–174 (1953); MR **14**, 1093] for which no assumptions are made concerning the behaviour of the semi-group in the neighbourhood of $\xi=0$. The author proposes such a new class (A) and gives a necessary and sufficient condition for a closed l.o. to be the infinitesimal generator of a semi-group of class (A). §5. Adjoint theory is developped here from a viewpoint somewhat different from Feller's [ibid. **57**, 287–308 (1953); MR **14**, 881]. Let U be a closed linear operator on X into X and let U^* be its adjoint. Let $\hat{U}$ be the restriction of U^* to the subspace $\hat{X}=$ the strong closure of $D(U^*)\subseteq X^*$, the adjoint space of X. Given a semi-group $\{T(\xi: U)\}$ of class (A), it is shown that the semi-group $\{S(\xi)\}$ generated by $\hat{U}$ is again of class (A) and $S(\xi)$ is the restriction of $T(\xi: U)$ to the domain $\hat{X}$. §6. Operational calculus and spectral theory associated with the semi-group $\{T(\xi: U)\}$.
K. Yosida (Tokyo).

Ehrenpreis, Leon. Solution of some problems of division. I. Division by a polynomial of derivation. Amer. J. Math. **76**, 883–903 (1954).

Ce travail est en rapport avec les recherches de Malgrange [C. R. Acad. Sci. Paris **237**, 1620–1622 (1953); MR **15**, 626], sur les équations aux dérivées partielles $PT=S$, où l'inconnue T et le terme indépendant S sont des distributions sur $\mathbf{R}^n$ et P est un polynôme de dérivation (à coefficients constants). Soit $\mathfrak{D}_F$ la limite projective (intersection) des espaces $\mathfrak{D}^s$ ($s=0, 1, \cdots$) des fonctions s fois continûment dérivables et à support compact dans $\mathbf{R}^n$, avec leurs topologies naturelles. (L'espace $\mathfrak{D}_F$ est, comme l'espace de Schwartz, formé par les fonctions indéfiniment dérivables à support compact, mais sa topologie est moins fine que celle de $\mathfrak{D}$; le dual de $\mathfrak{D}_F$ est l'espace des distributions d'ordre fini dans $\mathbf{R}^n$.) La transformation de Fourier est définie dans $\mathfrak{D}$ par la formule

$$\mathfrak{F}[f]=\int_{\mathbf{R}^n} f(x)e^{-ix\cdot z}, \quad \text{avec} \quad x\in\mathbf{R}^n, \quad z\in\mathbf{C}^n$$

On désigne par $\mathbf{D}$ l'ensemble des images $\mathfrak{F}[f]$ (fonctions entières de type exponentiel et à décroissance rapide) avec la topologie qui rend bicontinue l'application $\mathfrak{F}$ de $\mathfrak{D}$ sur $\mathbf{D}$, et par $\mathbf{D}_F$ le même ensemble avec la topologie image de celle de $\mathfrak{D}_F$ par $\mathfrak{F}$. La transformation de Fourier est définie dans $\mathfrak{D}'$ comme la transposée de $\mathfrak{F}^{-1}$. Après quelques théorèmes, on trouve le résultat suivant: quel que soit le polynôme de dérivation P, la correspondance $Pf\to f$ est une application linéaire continue de $P\mathfrak{D}_F$ dans $\mathfrak{D}_F$. On en déduit l'existence d'une solution $T\in\mathfrak{D}_F'$ pour l'équation $PT=S$, avec $S\in\mathfrak{D}_F'$. L'auteur généralise ces résultats pour l'espace des distributions à support compact (avec une topologie moins fine que celle de Schwartz) et pour le cas des systèmes d'équations aux dérivées partielles à coefficients constants.

J. Sebastião e Silva (Lisbonne).

Orlicz, W. On a class of operations over the space of continuous vector valued functions. Studia Math. **14** (1954), 285–297 (1955).

Orlicz, W. On a class of operations over the space of integrable functions. Studia Math. **14** (1954), 302–309 (1955).

These two papers contain parallel theorems about subsets of two large normed spaces, and give conditions on an operator U, carrying the large space into itself, sufficient that it carry the given subsets of interest into themselves.

In the first paper the basic space is the normed space $C(X)$ of continuous functions defined on a real interval I with values in a given Banach space X. The interesting subsets of $C(X)$ are the spaces $L_\omega(X)$ of functions satisfying a Lipschitz condition of order ω, where $\omega(u)$ is a function from non-negative reals to themselves such that $\lim_{u\to 0}\omega(u)=0$.

In the second paper the large space is the space of summable functions on I and the subsets of interest are the spaces L^M. [See Orlicz, Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. **1936**, 93–107 for more about these generalized L^p spaces.] *M. M. Day.*

Kalugina, E. P. The class L_Φ as a convex functional manifold. Dokl. Akad. Nauk SSSR (N.S.) **98**, 13–16 (1954). (Russian)

This paper deals primarily with Orlicz's condition Δ_s which makes the Orlicz space L_Φ^* into a linear space. [References: W. Orlicz, Bull. Internat. Acad. Polon. Sci. Lett. Ser. A. **1932**, 207–220; **1936**, 93–107; L. V. Kantorovič, Mat. Sb. N.S. **2(44)**, 121–168 (1937); L. V. Kantorovič, B. Z. Vulih, and A. G. Pinsker, Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 340; A. C. Zaanen, Linear analysis, North-Holland Publ. Co., Amsterdam, 1953; Ann. of Math. (2) **47**, 654–666 (1946); MR **15**, 878; **8**, 158; we shall use the nomenclature as found in the two above-cited books.] Theorem 1 gives four necessary and sufficient conditions equivalent to Δ_s in terms of a certain convergence concept in L_Φ^* and weak and order convergence in L_Φ^* (see Orlicz above, and the article by Kantorovič). Theorems 2 and 3 state that convex linear functionals in L_Φ^* are linear (where possible). Theorem 5 gives an integral representation of linear functionals on L_Φ^* which are "t-continuous" (see Kantorovič) which reduces to the one on p. 138 of Zaanen's book; here $g(x)$ is such that if $0<q<1$, then $qg(x)\in L_\Phi^*$ where Ψ is complementary to Φ. There are no proofs.

G. K. Kalisch (Minneapolis, Minn.).

Bochner, S. Positive zonal functions on spheres. Proc. Nat. Acad. Sci. U. S. A. **40**, 1141–1147 (1954).

Let $\gamma>0$ and consider the L_p-class with the norm $\|f\|=(\int_{-1}^1|f(x)|^p d\mu(x))^{1/p}$, where

$$d\mu(x)=\left(\int_{-1}^1(1-t^2)^{\gamma-1/2}dt\right)^{-1}(1-x^2)^{\gamma-1/2}dx,$$

the ordinary class $C[-1,1]$ and the ordinary class $V[-1,1]$ of σ-additive set functions $F(A)$. By orthogonalizing $1, x, x^2, \cdots$ relative to the measure $d\mu(x)$, we get a sequence of polynomials $P_0(x), P_1(x), P_2(x), \cdots$ under the normalization $P_n(0)=1$: we have in this case

$$\int_{-1}^1 P_n^2(x)d\mu(x)=\gamma(\gamma+n)^{-1}\Gamma(n+2\gamma)^{-1}\Gamma(2\gamma)n!.$$

Theorem 1 states that, if $f\in C$ or $f\in L_p$, then

$$f^r(x)=\sum\rho_n\lambda_n{}^r a_n P_n(x) \quad \left(a_n=\int_{-1}^1 f(x)P_n(x)d\mu(x)\right)$$

converges in norm to f as $r\to\infty$. Here $\sum\rho_n\lambda_n{}^r P_n(x)$ is the "Fourier expansion" of a polynomial $k^r(x)\geqq 0$ such that $\int_{-1}^1 k^r(x)d\mu(x)=1$ and $\lim_{r\to\infty}\int_{-\epsilon}^{\epsilon}k^r(x)d\mu(x)=1$ for any $\epsilon>0$. A sequence $\{c_n\}$, $n=0, 1, \cdots$, is called "positive definite" if $\sum\rho_n a_n c_n P_n(x)$ is $\geqq 0$ whenever $\sum\rho_n a_n P_n(x)$ is $\geqq 0$. Theorem 2 states that $\{c_n\}$ is positive definite if and only if $c_n=\int_{-1}^1 P_n(x)dH(x)$ with a non-negative $H\in V$. A sequence of continuous functions $\{c_n(t),\ 0\leqq t<\infty\}$ $(n=0, 1, \cdots)$ is called a "homogeneous stochastic process" if (i) for each fixed t_0, $\{c_n(t_0)\}$ is positive definite, (ii) $c_n(t+s)=c_n(t)c_n(s)$ and (iii) $c_0(t)=1$ and $c_n(0)=1$. Theorem 3 states that $\{c_n(t)\}$ is a homogeneous stochastic process if and only if there is in the half-open interval $[-1, 1)$ a σ-additive set function $G(A)\geqq 0$ for which $\int_1^{1-0}dG(x)<\infty$ and a constant $\sigma\geqq 0$ such that

$$c_n(t)=\exp\left(-t\sigma n(n+2\gamma) - t\int_1^{1-0}(1-y)^{-1}(1-P_n(y))dG(y)\right).$$

K. Yosida (Tokyo).

Lorch, E. R. On the volume of smooth convex bodies in Hilbert space. Math. Z. **61**, 391–407 (1955).

Let H denote a real separable Hilbert space. It is well-known that in the attempt to define for H an analogue of Lebesgue measure for E^n, some of the familiar properties of Lebesgue measure must be sacrificed. Some of the existing

translation-invariant measures for H may be briefly described as follows: Loewner [Ann. of Math. (2) 40, 816–833 (1939); MR 1, 48]—locally finite, but takes its values in a non-Archimedean ordered field; Oxtoby (for an arbitrary uncountable complete metric group) [Trans. Amer. Math. Soc. 60, 215–237 (1946); MR 8, 253]—real-valued, but not locally finite; Brownell (also in a more general setting) [Pacific J. Math. 2, 531–553 (1952); MR 14, 963]—real-valued and locally finite, but the sets of finite positive measure do not fill the space. The present paper approaches the problem from quite a different viewpoint, extending to H some analytic machinery developed earlier for E^n by the author [Trans. Amer. Math. Soc. 71, 243–266 (1951); Ann. Mat. Pura Appl. (4) 34, 105–112 (1953); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 25–29 (1954); MR 14, 678, 657; 16, 345] and indicating how it may be used to obtain for certain families of smooth convex bodies a real-valued volume invariant under certain families of linear transformations. The actual realization of this program is not carried far, so the paper presents not a fruitful theory of volume in Hilbert space, but rather the description of a direction in which such a theory can perhaps be developed.

It is observed that each convex body in H which satisfies certain smoothness and curvature conditions generates in a natural way a function $\mathfrak{G}$ associating with each nonzero x in H a positive definite self-adjoint linear transformation $\mathfrak{G}(x)$ of H onto H. This leads to the postulation of such a function (henceforth in this review called "admissible") having $\mathfrak{G}(x)-I$ completely continuous for x and subject to various other restrictions. For admissible $\mathfrak{G}$, the set $\mathfrak{K}=\{x: (\mathfrak{G}(x)x, x)\leqq 1\}$ is convex, $(\ ,\ )$ denoting the inner product. The set of all admissible functions is convex (useful for the theory of mixed volumes). With $D(x)=\prod_1^n (1+\alpha_x^{(i)})$, D is a bounded continuous function, $\{\alpha_x^{(i)}\}_1^n$ being the sequence of characteristic values of $\mathfrak{G}(x)-I$. For each admissible function $\mathfrak{G}$ there is a unique admissible function $\mathfrak{K}$ such that $\mathfrak{K}(\mathfrak{G}(x)x)=\mathfrak{G}^{-1}(x)$ for each x, and the polar body $\mathfrak{K}^*$ is defined to be $\{x^*: (\mathfrak{K}(x^*)x^*, x^*)\leqq 1\}$.

Now suppose $\mathfrak{G}_l$ is a family of admissible functions (where l ranges over some index set) and $\mathfrak{R}$ is a family of transformations of the unit sphere Ω onto itself. Let B be the smallest Banach lattice of bounded continuous real functions on Ω which includes all the functions D_l and is closed under the transformations in $\mathfrak{R}$. If it happens that B admits a unique $\mathfrak{R}$-invariant integral F (i.e., a positive $\mathfrak{R}$-invariant bounded linear functional F of norm 1), the author suggests the definition $V_n\mathfrak{K}_l^*=FD_l$ for volumes of the sets $\mathfrak{K}_l^*$. When $\mathfrak{R}$ is the class of all rotations in H, these ideas are shown to lead to the product of the semi-axes as a volume formula for a certain class of ellipsoids. (As described, the volume appears to depend on $\mathfrak{G}_l$ as well as $\mathfrak{K}_l^*$. However, there is a duality between $\mathfrak{G}$ and $\mathfrak{K}$ as defined above, and thus if the sets $\mathfrak{K}_l^*$ are suitably restricted, they generate admissible functions $\mathfrak{K}_l$ in a natural way, and then the dual functions $\mathfrak{G}_l$ can be employed in the above definition.)

V. L. Klee (Seattle, Wash.).

Bram, Joseph. Subnormal operators. Duke Math. J. 22, 75–94 (1955).

By a subnormal operator A on a Hilbert space $\mathfrak{H}$ the author means an operator having a normal extension (i.e. $\mathfrak{H}$ can be embedded in a Hilbert space $\mathfrak{K}$ on which there is a normal operator B whose restriction to $\mathfrak{H}$ is A). Such operators were introduced by Halmos [Summa Brasil. Math. 2, 125–134 (1950); MR 13, 359] (N.B.: Halmos uses "subnormal" in a different sense in this paper). The author sharpens a result of Halmos's, proving that A is subnormal on $\mathfrak{H}$ if and only if $\sum_{n,m=0}^{r} (A^n f_m, A^n f_m)\geqq 0$ for every finite set $f_0, f_1, \cdots, f_r$ of elements of $\mathfrak{H}$. He also shews that A is subnormal if and only if $T(s)=\exp(-sA^*)\cdot\exp(sA)$ is a positive definite operator function in the complex plane (i.e. $\sum_{i,j=1}^{n} (f_i, T(s_j-s_i)f_j)\geqq 0$ for any set of elements $f_1, \cdots, f_n$ and any set of complex numbers $s_1, \cdots, s_n$). The author discusses the relation between the spectrum of a subnormal operator A and that of its minimal normal extension B, paying particular attention to the case when $\mathfrak{K}$ is cyclic with respect to B (in this connection he obtains a result which appears to be at variance with an assertion of Hamburger's [Proc. London Math. Soc. (3) 1, 494–512 (1951), p. 512; MR 13, 661]), and considers the extension of operators commuting with A to operators commuting with B.

A. F. Ruston (Sheffield).

Brownell, F. H. Flows and noncommuting projections on Hilbert space. Pacific J. Math. 5, 1–16 (1955).

Let X be a Hilbert space and E a spectral measure on X over the Borel sets of the line. For $u_0 \in X$ and t real define a flow

$$U_t u_0 = u_t = (L)\int \exp(it\lambda)\,dE(\lambda)u_0.$$

Let P be an orthogonal projection that, for the sake of nontriviality, does not commute with E. The author considers the asymptotic behavior of $\|Pu_t\|^2$ as $t\to+\infty$ or $-\infty$. He obtains $\|Pu_t\|^2=(R)\int\int \exp(it[x-y])\,d\gamma(x,y)$, where γ is a finitely additive complex-valued set function defined on the field of plane sets generated by rectangles with Borel sides, $\gamma(A\times B)=(PE(A)u_0, E(B)u_0)$. Call P tractable if γ has a bounded countably additive Borel extension. (Sufficient for this is P compact.) In the tractable case these are equivalent: (i) $\|Pu_t\|^2\to C$ densely as $t\to+\infty$; (ii) same, but for $t\to-\infty$; (iii) γ is concentrated on the line $y=x$ and attributes to it the measure C. Two examples of nontractable pathologies conclude the discussion.

J. G. Wendel (Ann Arbor, Mich.).

* **Julia, Gaston. Sur un propriété caractéristique des produits de deux symétries.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 36–39. Academic Press Inc., New York, 1954. $9.00.

A symmetry S in a Hilbert space $\mathfrak{H}$ is a linear operator that is both unitary and Hermitian; S is then the reflection with respect to the subspace $V=\{x: Sx=x\}=(I+S)\mathfrak{H}$, whose orthogonal complement is

$$V'=\{x: Sx=-x\}=(I-S)\mathfrak{H}.$$

In the present paper the following result is proved. An operator U is the product of two symmetries if and only if there is a subspace V that reduces $U+U^*$ and such that $U-U^*$ permutes V with its orthogonal complement V'. As a corollary, every member of the group generated by two non-commuting symmetries is either a symmetry or the product of two symmetries.

F. Smithies.

Nakamura, Masahiro, and Turumaru, Takasi. Completely continuous operators with property F. Tôhoku Math. J. (2) 6, 174–176 (1954).

Let A and B be two completely continuous, normal operators on Hilbert space, and one wants conditions to

insure that A and B commute, and thus have a common spectral resolution of the identity. Property F is the condition that the point spectrum of $p(A, B)$, for every polynomial $p(x, y)$, is, with proper multiplicity, p(spectrum A, spectrum B); property L is the weaker condition that this holds only for $p(x, y)=\alpha x+y$ for all complex α. Although this paper proves another statement as well, its main result is that property F implies that A and B commute for Hermitian A and B. This theorem is included in an earlier result of Kaplansky [Pacific J. Math. 3, 721–724 (1953); MR 15, 440], which obtains the same conclusion from property L alone. However, the method of proof is different here. *F. H. Brownell* (Seattle, Wash.).

Taldykin, A. T. On the problem of existence of eigenvalues of linear operators. Dokl. Akad. Nauk SSSR (N.S.) **99**, 905–908 (1954). (Russian)

Let A and B be two operators in Hilbert space both of which are either bounded or with unique bounded reciprocals. Then if AB has an eigenvalue in its Fredholm circle, so has BA; and if the eigenvectors and auxiliary vectors corresponding to eigenvalues in the Fredholm circle form a complete system for AB, so do they for BA.

J. L. B. Cooper (Cardiff).

Müller, H. Über eine Klasse von Eigenwertaufgaben mit nichtlinearer Parameterabhängigkeit. Math. Nachr. **12**, 173–181 (1954).

The author considers the eigenvalue problem

$$\lambda^n x-\sum_{\nu=0}^{n}\lambda^{n-\nu}A_\nu x=0 \tag{1}$$

in Hilbert space where $A_0, A_1, \cdots, A_n$ are self-adjoint completely continuous operators, and $(A_0x, x)\leqq q(x, x)$ for all x, where $q<1$. The main result is as follows. Denote by K the set of complex numbers κ satisfying an equation of the form

$$\kappa^n(x, x)-\sum_{\nu=0}^{n}\kappa^{n-\nu}(A_\nu x, x)=0$$

for some $x\neq 0$, and let P be the intersection of K with the real axis. Then K is bounded, (1) has a real eigenvalue if and only if P is non-void, every isolated point of P is an eigenvalue of (1), and the lower (upper) bound of P is an eigenvalue of (1), provided that it is distinct from 0. Examples are given to illustrate various possible forms for the sets K and P. *F. Smithies* (Cambridge, England).

Livšic, M. S. On an inverse problem of the theory of operators. I. Dokl. Akad. Nauk SSSR (N.S.) **97**, 399–402 (1954). (Russian)

The author proves the following result concerning matrix operators in the Hilbert space (l^2). Let H_p be a Hermitian matrix, linear with respect to the parameter p for $p>0$, and of the form $H_p=H_0+cpJ_0$, where H_0 is the matrix of a bounded Hermitian operator, $c>0$ and

$$J_0=\begin{bmatrix}1&0&0&\cdots\\0&0&0&\cdots\\0&0&0&\cdots\\\cdot&\cdot&\cdot&\cdots\end{bmatrix}.$$

Suppose also that for each $p>0$ the matrix H_p has the unique eigenvalue $E_p=c(m^2c^2+p^2)^{1/2}$, and that its spectrum is simple. Then H_p is unitarily equivalent to the matrix mc^2T+cpJ_0, where T is the Jacobi matrix

$$\begin{bmatrix}0&2^{-1/2}&0&0&\cdots\\2^{-1/2}&0&\frac{1}{2}&0&\cdots\\0&\frac{1}{2}&0&\frac{1}{2}&\cdots\\0&0&\frac{1}{2}&0&\cdots\\\cdot&\cdot&\cdot&\cdot&\cdots\end{bmatrix}.$$

From this it follows that the spectrum of the 'rest energy' matrix H_0 is continuous and fills the closed interval $(-mc^2, mc^2)$. The proof is mainly a straightforward application of the theory of Jacobi matrices. *F. Smithies.*

Livšic, M. S. On an inverse problem of the theory of operators. II. Dokl. Akad. Nauk SSSR (N.S.) **97**, 589–592 (1954). (Russian)

In this note the author generalizes the result of note I (reviewed above) as follows. Let $H_p=A+pB$ be Hermitian and linear with respect to p for $p>0$, and suppose that for each $p>0$ the matrix H_p has the unique eigenvalue $E_p=c(m^2c^2+p^2)^{1/2}$, where $c>0$. Suppose also that B is the matrix of a completely continuous operator. Then H_p is unitarily equivalent to a matrix of the form

$$\begin{bmatrix}T_p&0&\cdots&0&0\\0&T_p&\cdots&0&0\\\cdot&\cdot&\cdots&\cdot&\cdot\\0&0&\cdots&T_p&0\\0&0&\cdots&0&C\end{bmatrix},$$

where (in the notation of I) $T_p=mc^2T+cpJ_0$, and C is a constant Hermitian matrix. In particular, if H_p has a simple spectrum, it is unitarily equivalent to T_p. Relations with the theory of the Dirac wave equation are indicated.

F. Smithies (Cambridge, England).

Brodskiĭ, M. S. The multiplication theorem for characteristic matrix-functions of linear operators. Dokl. Akad. Nauk SSSR (N.S.) **97**, 761–764 (1954). (Russian)

The author generalizes the notion of a characteristic matrix-function of a bounded linear operator A in a Hilbert space H as follows. Suppose that the operator

$$\Im(A)=i(A^*-A)$$

is of finite rank, and let E be a finite-dimensional subspace containing the range of $\Im(A)$. Let $e_1, e_2, \cdots, e_r$ be a complete orthonormal system of eigenvectors of $\Im(A)$ in E with eigenvalues $\omega_1, \omega_2, \cdots, \omega_r$ respectively. Let m, s and n be the numbers of positive, zero and negative eigenvalues respectively, and let $p\geqq m+s$, $q\geqq n+s$. Now let

$$W(\lambda)=I-i\Pi^*[((A-\lambda P_E)^{-1}e_\alpha, e_\beta)]\Pi J,$$

where

$$J=\begin{bmatrix}I_p&0\\0&-I_q\end{bmatrix}$$

and Π is any rectangular matrix having r linearly independent rows and $p+q$ columns, and such that $\Pi J\Pi^*=\Omega$, Ω being the diagonal matrix whose diagonal elements are $\omega_1, \omega_2, \cdots, \omega_r$. Then $W(\lambda)$ is said to be a characteristic matrix-function of the operator A.

Let H_0 be a closed subspace of H with projector P_0; let A_0 be defined in H_0 by $A_0f=P_0Af$, and let $E_0=P_0E$. The author shows how a characteristic matrix-function $W_0(\lambda)$ for A_0 can be derived from $W(\lambda)$, and he calls $W_0(\lambda)$ the projection of $W(\lambda)$ on H_0. Now let H_1 be an invariant subspace of A, and let H_2 be its orthogonal complement. Let

$W_1(\lambda)$ and $W_2(\lambda)$ be the projections of $W(\lambda)$ on H_1 and H_2 respectively. The author shows that $W(\lambda)=W_2(\lambda)W_1(\lambda)$. Finally, a converse theorem is stated; this concerns the construction of A from A_1 and A_2. *F. Smithies.*

***Sz.-Nagy, Béla. Prolongements des transformations de l'espace de Hilbert qui sortent de cet espace. Appendice au livre "Leçons d'analyse fonctionnelle" par F. Riesz et B. Sz.-Nagy.** Akadémiai Kiadó, Budapest, 1955. 36 pp.

Let $H\subseteq\mathfrak{H}$ be Hilbert spaces, and let P be the orthogonal projection of $\mathfrak{H}$ on H. A bounded linear operator T from H into H is called the projection of such an operator S from $\mathfrak{H}$ into $\mathfrak{H}$ if $Tx=PSx$ for each x in H. A number of theorems have been proved asserting that an operator, or family of operators, of some kind is the projection of better operators in some superspace. For example, Halmos [Summa Brasil. Math. **2**, 125–134 (1950); MR **13**, 359] has characterized the projections of normal operators. Neumark [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **41**, 359–361 (1943); MR **6**, 71] showed that every generalized resolution of the identity is the projection of a true resolution of the identity.

In this paper the author proves a theorem from which half a dozen such results can be derived by shrewd substitutions. Define a *-semi-group to be a semi-group in which there is an involutory antiautomorphism $\mu\rightarrow\mu^*$; a useful example is a group with μ^* set equal to μ^{-1}. The principal theorem of this paper describes properties of a function $\mu\rightarrow T_\mu$ from a *-semi-group Γ to linear operators over H which are necessary and sufficient that there exist a superspace $\mathfrak{H}$ and a *-representation $\mu\rightarrow D_\mu$ of Γ over $\mathfrak{H}$ such that for each $\mu T_\mu=PD_\mu$. Some of these conditions resemble those of Gel'fand and Raĭkov [Mat. Sb. N.S. **13**(**55**), 301–316 (1943); MR **6**, 147] on the representation of functions of positive type on a group Γ. *M. M. Day* (Urbana, Ill.).

***Riesz, Frédéric, et Sz.-Nagy, Béla. Leçons d'analyse fonctionnelle. 3ème éd.** Gauthier-Villars, Paris; Akadémiai Kiadó, Budapest, 1955. viii+488 pp. 3000 francs.

Aside from minor changes, this edition differs from the last [1953; MR **15**, 132] in the addition of the Appendix reviewed above.

Calculus of Variations

***Morse, Marston. Introduction to analysis in the large. 2d ed.** The Institute for Advanced Study, Princeton, N. J., 1951. 160+31 pp. (mimeographed). $3.60.

These lectures give a self-contained development of critical-point theory and its application to the calculus of variations in the non-degenerate case. In this application, a critical point becomes an extremal arc joining two points A and B of a manifold and non-degeneracy amounts to the assumption that B is not conjugate to A on any such extremal. In a sense of density, this theory is general.

The first six sections are concerned with quadratic analysis, the index of the second variation based on an extremal arc and its evaluation in terms of the number of conjugate points and the characteristic roots. A proof of the author's generalizations of the Sturm separation theorem is included.

The next eighteen sections develop properties of the space W of arcs joining two points of a manifold in relation to an integral J defining a parametric problem of the calculus of variations and properties of the function defined on W by J. The objective is to make the requisite preparation for application to the given setting of the author's critical-point theory of a positive lower semi-continuous non-degenerate function on a metric space as set forth in an earlier paper [Fund. Math. **35**, 47–78 (1948); MR **10**, 391]. There are numerous technical difficulties to be overcome: reduction of the neighborhood of an arc to a single coordinate system; reduction to a non-parametric integral; existence of local fields and absolute minima; introduction of a metric in the space of arcs; proof of the lower semi-continuity of the integral as a function of arcs; proof of the existence of fundamental deformations.

The remaining paragraphs follow to a large extent the paper cited above. Additional results include a determination of the Betti numbers of the space of paths on any closed orientable surface of genus greater than zero and application of the critical-point relations to the normals from a point to a manifold.

There are two appendices concerned with material needed in the text. The first is on parameterization of curves [cf. Morse, Bull. Amer. Math. Soc. **42**, 915–922 (1936)] and the second on singular homology theory [cf. Eilenberg, Ann. of Math. (2) **45**, 407–447 (1944); MR **6**, 96].

G. A. Hedlund (New Haven, Conn.).

Stampacchia, Guido. Problemi variazionali per gli integrali multipli in forma non parametrica. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 19–29 (1954).

An exposition of the recent history of the nonparametric minimization problem in the calculus of variations.

J. M. Danskin (Washington, D. C.).

Grebenyuk, D. On the minimum of certain integrals. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 111–118 (1949). (Russian)

From the author's introduction: "Let the function $f(x)$ be defined in such a way as to satisfy the following conditions:

$$(1)\quad \int_a^b \pi(x)f(x)[x^kA(x^i)-A(x^k)x^i]dx=D(x^i,x^k) \qquad (i=0,\cdots,n),$$

where k is some fixed one of the numbers $0,\cdots,n$,

$$A(g)=\begin{vmatrix}\omega(g) & 1\\ \omega_1(g) & 1\end{vmatrix},\quad D(g,g_1)=\begin{vmatrix}\omega(g) & \omega(g_1)\\ \omega_1(g) & \omega_1(g_1)\end{vmatrix},$$

with

$$g(x)=g_0x^n+g_1x^{n-1}+\cdots+g_n,$$
$$g_i(x)=g_0{}^ix^n+g_1{}^ix^{n-1}+\cdots+g_n{}^i$$

two arbitrary polynomials of degree $\leqq n$ and

$$\omega(g)=\alpha_0g_0+\alpha_1g_1+\cdots+\alpha_ng_n,$$
$$\omega_1(g)=\beta_0g_0+\beta_1g_1+\cdots+\beta_ng_n,$$

where the α_i and β_i are two systems of given real numbers. It is required to determine the minimum of the integral

$$\int_a^b \pi(x)\,|f(x)|\,dx,$$

where $\pi(x)$ is a given integrable function, not vanishing on the segment $[a,b]$, and $f(x)$ is subjected to conditions (1), where not all of the $D(x^i,x^k)$ are zero simultaneously, since then the minimum sought would obviously be zero."

The author finds the form of the solution too complicated to give here. *J. M. Danskin* (Washington, D. C.).

Koschmieder, Lothar. Some isoperimetric problems of geometrical and mechanical interest. Tecnica. Rev. Fac. Ci. Ex. Tec. Univ. Nac. Tucuman 1, 80–90 (1951). (Spanish. English summary)

The author discusses the isoperimetric problem of maximizing $I=\int_{x_0}^{x} y^n dx$ in a class of admissible arcs joining a variable point (x_0, y_0) in the half-plane $y>0$ to a fixed point (x_1, y_1) in this half-plane, and giving a fixed value to $J=\int_{x_0}^{x} y^n[1+(dy/dx)^2]^{1/2}dx$, in the following cases: $m=2$, $n=1$; $m=1$, $n=2$; $m=3$, $n=1$; $m=1$, $n=3$. For the individual problems the equations of the extremals are given, and conditions for a strong relative maximum are derived by the consideration of certain types of fields; geometric or mechanical characterizations of the problems are provided through the interpretation of I and J as suitable moments.

W. T. Reid (Evanston, Ill.).

Theory of Probability

***Gnedenko, B. W., und Chintschin, A. J. Elementare Einführung in die Wahrscheinlichkeitsrechnung.** Deutscher Verlag der Wissenschaften, Berlin, 1955. 136 pp.

Translation of Gnedenko and Hinčin, Ėlementarnoe vvedenie v teoriyu veroyatnostei, 3d ed. [Gostehizdat, Moscow, 1952; MR 14, 293].

Neyman, Jerzy. The problem of inductive inference. Comm. Pure Appl. Math. 8, 13–45 (1955).

The paper deals with general ideas illustrated by means of a problem from modern physics (the homogeneity of neutral V-particles). There is also a two-page summary of some recent work on mathematical statistics.

The author's views on the theory of probability and inductive inference are what the reviewer would call "orthodox", with the modification that there is emphasis on the making of decisions (so that some people would prefer such names as "theory of rational behaviour" and "theory of inductive behaviour"). This emphasis occurs only in the suggestions for the application of the abstract theory, not in the axiomatic scheme, which, being Kolmogoroff's, makes no reference to utilities. The insistence on the distinction between an abstract theory and its practical application is not very controversial to-day among applied mathematicians with philosophical leanings. Each practical application uses a judgment that some model belonging to the abstract theory is adequate for the application. An explicit interpretation of the word "adequate" is not given. The author would perhaps not object to a subjectivistic interpretation of the word, provided that the models themselves contained only probabilities interpretable by long-range frequencies. The purpose of this restriction is to make indisputable probability statements in each application. The author does not mention that the restriction obtains precision at the cost of a loss of information.

The V-particle problem illustrates the well known statistical technique of E. S. Pearson and the author [Neyman, Philos. Trans. Roy. Soc. London. Ser. A. 231, 289–337 (1933); 236, 333–380 (1937)]. *I. J. Good.*

Ville, J. A. Leçons sur quelques aspects nouveaux de la théorie des probabilités. Ann. Inst. H. Poincaré 14, 61–143 (1954).

There are three quite separate chapters, each with a great variety of topics. Much of the material is strictly original; some, though novel in detail, is expository of known facts.

Chapter I. Probability and frequency. The central theme of the chapter is this. A (potentially infinite) table of independent random binary digits (0 and 1 equally probable) can be coded to yield a table of random numbers i $(=1, \cdots, n)$ independent with arbitrary probabilities p_i. The expected number of binary digits consumed in producing an i is at least $-(\log 2)^{-1}\sum p_j \log p_j$, and this lower bound can be approached if sufficiently many i's are called for. These main results are close to, if not identical with, results obtained by Shannon [Bell System Tech. J. 27, 379–423, 623–656 (1948) [MR 10, 133], reprinted in Shannon and Weaver, The mathematical theory of communication, Univ. of Illinois Press, 1949 [MR 11, 258]] but many details, techniques, and ramifications in the chapter seem new.

This chapter also proposes a test of a given multinomial distribution against all alternatives on the basis of a sample of 1. This test is a generalization of a test suggested by M. S. Bartlett [Biometrika 39, 228–237 (1952); MR 14, 666].

Chapter II. Theory of games. Much familiar material is elegantly reviewed and some new topics are introduced. Two of these will be mentioned.

It is well known that every position in chess (including of course the initial position) is a chess puzzle in the sense that one player or the other can positively force the checkmate of the other or else both players can force a draw. There therefore exists an automaton, in principle, that can win or force a draw if its opponent ever provides an opportunity. Any such automaton plays perfect chess in one sense, but some of them play very poor chess. Some, for example, move indiscriminately when in a losing position, including those losing positions that no human could hope to recognize as such. A way of avoiding this deficiency is suggested.

Turning to the topic of many-person games, the theory of coalitions takes on a new form if it is postulated that collaborators, in effecting their mixed strategies, must refer to independent randomizing devices so that they cannot employ arbitrary joint mixed stragegies.

Chapter III. Presents many aspects of the theory of communication, more or less closely connected with Shannon's theory mentioned in connection with Chapter I.

L. J. Savage (Chicago, Ill.).

Andreoli, Giulio. Sul concetto di frequenza relativa: sue connessioni con certe serie di potenze e certe algebre di Boole. Giorn. Mat. Battaglini (5) 2(82), 379–404 (1954).

This paper is a study of 1, 0 sequences. The author considers operations (essentially those defined by von Mises) on such sequences and the corresponding operations induced on the generating functions of the sequences. He obtains an integral (the Parseval integral) in terms of which he expresses Boolean operations on the generating functions. He points out that associated with a 1, 0 sequence is the sequence of relative frequencies which in turn can be converted into various 1, 0 sequences and that this process can be repeated. *A. H. Copeland, Sr.* (Ann Arbor, Mich.).

Chu, John T. On bounds for the normal integral. Biometrika 42, 263–265 (1955).

Bass, Jean. Sur la compatibilité des fonctions de répartition. C. R. Acad. Sci. Paris 240, 839–841 (1955).

The author asks, in what circumstance does there exist a 3-dimensional distribution-function $F(\cdot, \cdot, \cdot)$ having given marginal distributions $F(\infty, \cdot, \cdot)$, $F(\cdot, \infty, \cdot)$ and $F(\cdot, \cdot, \infty)$? He states a condition of compatibility for the margins; it is not clear from the paper whether this is necessary and sufficient, or merely necessary. The condition leads to the

following subsidiary problem. Given three 1-dimensional distributions, let X, Y and Z be random variables (independent or not) having the given distributions and let ξ, η, ζ be their three covariances; it is required to characterise the set Ω consisting of all possible triads (ξ, η, ζ). The author remarks that Ω is convex and lists a number of further properties. *D. G. Kendall* (Oxford).

Zubrzycki, S. Les inégalités entre les moments des variables aléatoires équivalentes. Studia Math. **14** (1954), 232–242 (1955).

Proofs of theorems announced in a previous paper [Bull. Acad. Polon. Sci. Cl. III. **2**, 59–61 (1954); MR **16**, 267]. As application, a statistical law of E. B. Levy [New Zealand J. Agric. **46** (1933)], relating to the random throwing of a marked ruler on a plane lattice, is formalized and justified. *J. L. Doob* (Urbana, Ill.).

Fisz, M. A generalization of a theorem of Khintchin. Studia Math. **14** (1954), 310–313 (1955).

Two (multidimensional) probability functions are of the same type if one can be obtained from the other by a real linear transformation whose homogeneous part is non-singular. If two sequences of non-singular probability functions of the same type converge to non-singular probability functions, then the latter must be of the same type. *K. L. Chung* (Syracuse, N. Y.).

Korolyuk, V. S. Asymptotic expansions for the criteria of fit of A. N. Kolmogorov and N. V. Smirnov. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 103–124 (1955). (Russian)

Let $F(t)$ be a continuous distribution function and $S_n(t)$ and $T_m(t)$ be empiric distributions of sample size n and m of independent random variables each having distribution $F(t)$. Let

$$\Phi_n(z)=P\{n^{1/2}\sup_{-\infty<t<\infty}|S_n(t)-F(t)|<z\},$$

$$\Phi_{nm}(z)=P\left\{\left(\frac{nm}{n+m}\right)^{1/2}\sup_{-\infty<t<\infty}|T_m(t)-S_n(t)|<z\right\},$$

$$\Phi_n^+(z)=P\{n^{1/2}\sup_{-\infty<t<\infty}[F(t)-S_n(t)]<z\},$$

$$\Phi^+_{nm}(z)=P\left\{\left(\frac{nm}{n+m}\right)^{1/2}\sup_{-\infty<t<\infty}[T_m(t)-S_n(t)]<z\right\}.$$

The author computes the asymptotic expansions of the above functions as $n, m\to\infty$ to terms of order $1/n$. The principal terms are well known results due to Kolmogoroff and Smirnov and the $n^{-1/2}$ term of Φ_n^+ was also given by Smirnov. The special cases of Φ_{nm} and Φ^+_{nm} where $n=m$ were treated by Gnedenko [Dokl. Akad. Nauk SSSR (N.S.) **82**, 661–663 (1952); slightly revised results in Math. Nachr. **12**, 29–63 (1954); MR **13**, 760; **16**, 498]. The author's results are incompatible with these special results in both cases. *J. Blackman* (Syracuse, N. Y.).

Fisz, M. The limiting distributions of the multinomial distribution. Studia Math. **14** (1954), 272–275 (1955).

Let

$$P(x_{n1}=k_1,\cdots,x_{nr}=k_r)=\frac{n!}{k_1!\cdots k_r!}p_1^{k_1}\cdots p_r^{k_r}.$$

If the random vector $(A_{n1}x_{n1},\cdots,A_{nr}x_{nr})$, where the A's are positive and the B's are real, converge in distribution to a non-singular $(r-1)$-dimensional vector, the latter must be of the form (ξ,η), where ξ is j-dimensional Poisson variable, $0\leqq j\leqq r-1$, η is a normal variable, and ξ and η are independent. *K. L. Chung* (Syracuse, N. Y.).

Sparre Andersen, Erik. On the fluctuations of sums of random variables. II. Math. Scand. **2**, 195–223 (1954).

The author continues his previous study [Math. Scand. **1**, 263–285 (1953); MR **15**, 444]. We refer to our previous review for terminology. In case (b) explicit expressions for $\Pr(K_n=m)$ and $\Pr(K_n=m\,|\,S_{n+1}>0)$ are obtained in terms of $a_i=\Pr(S_i>0)$. The results are quite simple if all a_i are equal and lend to sharp approximations. If $a_n\to a$, $0<a<1$, the limiting distribution of K_n/n is obtained which becomes for $a=\frac{1}{2}$ the arcsin law. A new random variable H_n is studied which is the number of values i, $1\leqq i\leqq n-1$, for which S_i coincides with the "largest convex minorant" of the sequence $S_0, S_1, \cdots, S_n$. If the common distribution F in case (b) is continuous, then the distribution of H_n does not depend on F. Various generating functions relating to K_n and H_n are given and there are more complicated results which cannot be readily summarized. *K. L. Chung.*

Sparre Andersen, Erik. Remarks to the paper: On the fluctuations of sums of random variables. Math. Scand. **2**, 193–194 (1954).

An error noted by the reviewer [MR **15**, 444] and a gap in a proof are fixed. *K. L. Chung* (Syracuse, N. Y.).

Fuchs, Aimé. Sur un théorème de N. Wiener. C. R. Acad. Sci. Paris **240**, 1396–1398 (1955).

Let R be the real numbers, let I be a compact interval, let $a(t,x)\geqq 0$ and $b(t,x)$ be real and measurable on $I\subset R$, and let $(\Xi(t): t\in I)$ be a Brownian motion. The author states (1) that a stochastic process $(x(t): t\in I)$, satisfying the stochastic differential equation

$$dx(t)=b(t,x(t))dt+a(t,x(t))^{1/2}d\Xi(t)$$

(in some unspecified sense) is necessarily a Markov process; (2) that $x(t)$ is continuous on I with probability 1, provided a and b are bounded and

$$(*)\quad E((a(u,x(u))^{1/2}-a(u,x(t))^{1/2})^2\,|\,x(t)=x)\leqq \text{const.}\times(u-t)^{\epsilon}\quad(t<u)$$

for some positive ϵ; and (3) that

$$|x(t+h)-x(t)|<c(2h\log h^{-1})^{1/2}$$

for sufficiently small h, uniformly on I, with probability 1, provided (*) is satisfied for some $\epsilon>1$ and c is taken greater than $\sup_{I\times R}a(t,x)^{1/2}$. *H. P. McKean Jr.*

Fuchs, Aimé. Sur certains opérateurs linéaires associés aux processus réels de Markoff. C. R. Acad. Sci. Paris **240**, 1506–1508 (1955).

Let R be the real numbers and let $(x(t): t>0)$ be a Markov process with state space R and transition probabilities

$$(1)\qquad F(t,x;\tau,E)=P(x(\tau)\in E\,|\,x(t)=x)\quad(t<\tau),$$

E being a Borel set in R. The author supposes (but neglects to state) that (1) is Borel measurable as a function of x for each choice of (t,τ,E), that it is a Borel probability measure as a function of E for each choice of (t,x,τ), and that the Chapman-Kolmogorov identity is satisfied.

Let M ($\bar{M}$) be the Banach space of real, finite, Borel measures (real, bounded, Borel functions) on R provided with the customary norm, and consider the operators "en arrière",

$$T_{t,\tau}:\quad \varphi\in M\to\int_R\varphi(dx)F(t,x;\tau,\cdot)\quad(t<\tau),$$

and the operators "en avant",

$$T_{\tau,t}: \quad \bar{\varphi} \in \bar{M} \to \int_R \bar{\varphi}(x) F(t, \cdot; \tau, dx) \quad (t < \tau).$$

Suppose

$$(2) \qquad \lim_{\lambda \downarrow 0} h^{-1}(T_{t-\lambda,t} - 1) = A(t) = \lim_{\lambda \downarrow 0} h^{-1}(T_{t,t+\lambda} - 1)$$

exists, the limits being taken in the uniform operator topology over M. Then (1) satisfies the generalized Kolmogorov equations,

$$(\partial/\partial\tau - A(\tau)) F(t, x; \tau, \cdot) = 0 = (\partial/\partial t + \bar{A}(t)) F(t, \cdot; \tau, E),$$

$\bar{A}(t)$ being the appropriate dual to $A(t)$. A necessary and sufficient condition that the left-hand limit in (2) should exist is given.

Remark: the operators "en arrière" ("en avant") are usually called the forward (backward) operators.

H. P. McKean Jr. (Princeton, N. J.).

Černov, L. A. Correlation of amplitude and phase fluctuations in the propagation of a wave in a statistically nonhomogeneous medium. Dokl. Akad. Nauk SSSR (N.S.) **98**, 953–956 (1954). (Russian)

L'auteur étudie les variations irrégulières des différents facteurs d'un milieu quelconque. Lorsqu'une onde se propage, les irrégularités introduisent une certaine dispersion de l'onde. Les ondes dispersées avec l'onde primaire provoquent des fluctuations de l'amplitude et de la phase du champ résultant.

Il s'agit de trouver la corrélation entre les fluctuations de l'amplitude et de la phase au point de réception ainsi que la corrélation de l'amplitude ou de la phase entre les différents points de réception, en supposant le milieu isotrope sans variations régulières. On suppose en outre que les variations aléatoires de coefficient de réfraction du milieu sont très petites par rapport à sa valeur moyenne et leurs valeurs très grandes par rapport à la longueur d'onde. En se basant sur les recherches de A. M. Obuhov [Izv. Akad. Nauk SSSR. Ser. Geofiz. **1953**, 155–165; MR **15**, 1002] l'auteur arrive à la conclusion que dans le cas des variations irrégulières à grande échelle les corrélations entre l'amplitude et la phase, ainsi que les corrélations de l'amplitude ou de la phase en différents points de réception, s'étendent à une distance de même ordre de grandeur que les corrélations entre les irrégularités aléatoires du milieu lui-même.

M. Kiveliovitch (Paris).

***Chernov, L. A. Correlation of the amplitudes and phases in wave propagation in statistically-inhomogeneous media.** Morris D. Friedman, Russian Translation, Two Pine Street, West Concord, Mass., 1954. 7 pp. (mimeographed). $3.50.

Translation of the paper reviewed above.

Bertaut, Félix. Statistique des fonctions complexes. Application à la cristallographie. C. R. Acad. Sci. Paris **240**, 938–940 (1955).

The author continues his two previous notes [same C. R. **240**, 152–154, 272–274 (1955); MR **16**, 493]. The content is about the same. *H. P. McKean, Jr.* (Princeton, N. J.).

***Kronheim, Helga. Beitraege zur technischen Behandlung der anormalen Risiken in der Lebensversicherung.** Dissertation, Universität Zürich, 1952. 67 pp.

An expository account. *H. Seal* (New York, N. Y.).

Mathematical Statistics

Dumas, Maurice. Les raisonnements de la statistique. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 158–162 (1954).

Barberi, Benedetto. Statistics and the theory of probability. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 42–59 (1954).

Gini, Corrado. On some symbols that may be usefully employed in statistics. Bull. Inst. Internat. Statist. **23**, part II, 249–282 (1951).

de Finetti, Bruno. Media di decisioni e media di opinioni. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 144–157 (1954).

De Finetti, Bruno. Concetti sul "comportamento induttivo" illustrati su di un esempio. Statistica, Bologna **14**, 350–378 (1954).

To illustrate the conceptual issues in various theories of statistical decision, a single example, chosen for simplicity and convenience, is intensively studied. This is an amplification of the author's earlier study of the same example [in the paper listed above]. The example consists of the problem of deciding, for a one dollar stake, whether an unknown probability is 1/3 or 2/3, with observations available one by one at 8/549 $(=2^3/3[2^7+7\times 2^3-1])$ dollars each. Extensive tables pertaining to this example are given, which are of potential value in teaching and research.

L. J. Savage (Chicago, Ill.).

Haldane, J. B. S. Substitutes for χ^2. Biometrika **42**, 265–266 (1955).

Greenwood, Robert E. Coupon collector's test for random digits. Math. Tables Aids Comput. **9**, 1–5 (1955).

The author extends the tabulated values of the probability distribution of the number of random digits n $(10 \leqq n \leqq 35)$ needed to include the complete set of all 10 digits. Approximate values are given also for $36 \leqq n \leqq 75$. *M. Muller.*

Dunnett, C. W., and Sobel, M. Approximations to the probability integral and certain percentage points of a multivariate analogue of Student's t-distribution. Biometrika **42**, 258–260 (1955).

For the multivariate analogue of Student's t-distribution, discussed by the authors in a previous paper [Biometrika, **41**, 153–169 (1954); MR **15**, 885], approximations (which are also lower bounds) to the probability integral are given for certain special cases when the number of variates is greater than two. *L. Weiss* (Charlottesville, Va.).

Moriguti, Sigeiti. Bounds for second moments of the sample range. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 57–64 (1954).

"This is an extension of a previous study" [Ann. Math. Statist. **22**, 523–536 (1951); MR **13**, 570]. Let X be a random variable symmetrically distributed (about 0, say) with variance $\sigma^2 < \infty$, and let R be the range in a sample of n independent observations of X. Let $U_n = \sup \delta(R^2)/\sigma^2$ and let V_n and C_n denote the infima of the variance of R/σ and of the coefficient of variation of R, respectively. Assuming tacitly that these extreme values are attained, the author applies the calculus of variations to reduce the determina-

tion of U_n, V_n, and C_n to that of characteristic values of certain linear integral equations. When n is even, the kernels of these equations are degenerate, and he finds that $1+C_n^2$ equals a quotient of determinants (of orders $n-3$ and $n-2$) whose elements can be evaluated by elementary quadratures: U_n and V_n, however, are only obtained as roots of transcendental equations which appear to be intractable except for quite small values of n. For $n=3$ a special argument is used. He gives the numerical values of C_n for $n=2, 3, 4, 6, 8$, and of U_n and V_n for $n=2, 3, 4$: for example, $C_3=0.0784$, $U_4=5.05$, $V_4=0.3591$. Presumably the suprema corresponding to V_n and C_n are not attained.
H. P. Mulholland (Birmingham).

Thomson, George W. Bounds for the ratio of range to standard deviation. Biometrika **42**, 268–269 (1955).

Chu, John T. On the distribution of the sample median. Ann. Math. Statist. **26**, 112–116 (1955).

Let $\tilde{x}$ be the median of a sample of $(2n+1)$ observations on a chance variable X having f and F as the probability density function (pdf) and cumulative distribution function (cdf) respectively and a unique median ξ. The distribution of $\tilde{x}$, under certain conditions [H. Cramer, Mathematical methods of statistics, Princeton, 1946, p. 369; MR **8**, 39], is known to be asymptotically normal (ξ, σ_n^2), where $\sigma_n^2=1/4[f(\xi)]^2(2n+1)$. In this paper upper and lower bounds are obtained for the cdf of $\tilde{x}$ and it is shown that if X is normally distributed, then the distribution of $\tilde{x}$ tends "rapidly" to normality. The cases when X has rectangular or Laplace distribution are also discussed where the approach to normality is found to be "much slower". It has been pointed out that the asymptotic normality of $\tilde{x}$ when X has a Laplace distribution cannot be deduced from the general theorem referred to above since it requires that $f'(\xi)$ be continuous, whereas for a Laplace distribution $f'(\xi)$ does not exist. *O. P. Aggarwal* (Seattle, Wash.).

Rao, C. Radhakrishna. An asymptotic expansion of the distribution of Wilks' Λ criterion. Bull. Inst. Internat. Statist. **23**, part II, 177–180 (1951).

The author derives an expansion using the distribution function of the beta distribution. The first term provides a powerful approximation, the second term being of order m^{-4}, where m has the same order as the sample size. The results have been incorporated in pp. 260–262 of C. Radhakrishna Rao, Advanced statistical methods in biometric research [Wiley, New York, 1952; MR **14**, 388]. *S. Kullback.*

Rios, Sixto. Comparison of random samples with and without replacement. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 98–102 (1954). (Spanish. French and English summaries)

The author equates the sampling variance of the average in a sample of size n, with replacement, and a sample of size n', without replacement, from a finite population of size N, and shows that $n'=nN/(n+N-1)$. He then shows that the expected number of distinct elements in the sample with replacement is $\geqq n'$, which shows the advantage of the sample without replacement. *S. Kullback.*

Varma, K. Bhaskara. On the exact distribution of Wilks' L_{mvo} and L_{vo} criteria. Bull. Inst. Internat. Statist. **23**, part II, 181–214 (1951).

The author uses Mellin's transform to develop a series expansion for the exact distribution of the L_{mvo} and L_{vo} criteria of Wilks [Ann. Math. Statist. **17**, 257–281 (1946); MR **8**, 162]. The 5% and 1% points of the distributions have been tabulated for k variates, $k=4, 5, 6, 7$, and various values of sample size n. These tabulated values have been used to examine the approximations suggested by Wilks and by Tukey and Wilks [ibid. **17**, 318–324 (1946); MR **8**, 162]. An example is given applying the results to a 4-variate problem on family budget data. *S. Kullback.*

Roy, J. The distribution of certain likelihood criteria useful in multivariate analysis. Bull. Inst. Internat. Statist. **23**, part II, 219–230 (1951).

The author uses the technique of introducing adjustable parameters to derive series expansions of the distribution of a number of likelihood criteria whose probability densities are such that the rth moment is given by a product of a ratio of gamma functions. Cases which are particularly considered are the L_{mvo} and L_{vo} criteria of Wilks [Ann. Math. Statist. **17**, 257–281 (1946); MR **8**, 162] and the bipolarity criterion of Votaw [ibid. **19**, 447–473 (1948); MR **10**, 387].
S. Kullback (Washington, D. C.).

Green, J. R. A confidence interval for variance components. Ann. Math. Statist. **25**, 671–686 (1954).

Given two statistics M_1 and M_2, which are independently distributed as $\sigma_1^2\chi^2/r_1$ and $\sigma_2^2\chi^2/r_2$ with r_1 and r_2 degrees of freedom, respectively, the problem is to find confidence limits for $\sigma_1^2-\sigma_2^2$, σ_1^2 and σ_2^2 being unknown. This is equivalent to finding a function f such that

$$\Pr[M_1/(\sigma_1^2-\sigma_2^2) \leqq f(M_2/(\sigma_1^2-\sigma_2^2))]=\alpha \quad (0<\alpha<1),$$

where α is given and f is independent of nuisance parameters. The author obtains a function which approximates f and discusses the tabulation. *R. P. Peterson.*

Lord, Frederic M. Sampling fluctuations resulting from the sampling of test items. Psychometrika **20**, 1–22 (1955).

When n dichotomic test items, labelled with $i=1, \cdots, n$, are administered to N examinees, labelled with $a=1, \cdots, N$, the outcome constitutes an n by N matrix $\{x_{ia}\}$ of 0, 1-variables. Various statistics based on this matrix are of interest: the average item difficulty $x_{..}$, the score variance over examinees $s^2=\sum(x_{.a}-x_{..})^2/N$, the reliability coefficient defined as a certain linear function of $\sum\sum(x_{ia}-x_{i.})^2/s^2$, etc. The author considers the examinees as fixed, the items as a random sample from a population of items, and derives formulas for the standard errors of the statistics mentioned, replacing the item population parameters by estimates based on the sample. He also derives certain formulas for the case that both items and examinees are sampled.
G. Elfving (Helsingfors).

Sen, A. R. On the estimate of the variance in sampling with varying probabilities. J. Indian Soc. Agric. Statist. **5**, 119–127 (1953).

A slight extension of a result obtained by D. G. Horvitz and D. J. Thompson [J. Amer. Statist. Assoc. **47**, 663–685 (1952); MR **14**, 777] for determining an estimator of the variance of the population total of a finite universe. To avoid having possible negative estimates for a variance the author suggests the obvious device of using a truncated but biased estimator, i.e., if a negative value is obtained, replace this value by zero. The paper concludes by considering a numerical example and a heuristic discussion on efficiency of the methods considered. *M. Muller* (Ithaca, N. Y.).

Chanda, K. C. Comparative efficiencies of L-test and Pitman's test in testing for equality of variances. Bull. Inst. Internat. Statist. **23**, part II, 215–218 (1951).

Cox, D. R., and Stuart, A. Some quick sign tests for trend in location and dispersion. Biometrika **42**, 80–95 (1955).

Let $y_i=\alpha+i\Delta+\epsilon_i$ where the ϵ_i are independently and identically distributed variables. Let $h_{ij}=1$ or 0 depending on whether $y_i>y_j$ or $y_i<y_j$. The paper discusses tests of the hypothesis $\Delta=0$ against the alternative $\Delta>0$ based on statistics of the type $\sum w_{ij}h_{ij}$ where the w_{ij} are weights and summation is over independent pairs of observations only. The asymptotic relative efficiency of several such tests is compared to that of rank correlation tests and the best parametric test (assuming normality of ϵ_i). For one of the tests, the small sample power is also investigated. By letting y_i be the range of groups of k successive observations, above sign tests can be used to test for trend in dispersion. Suggestions for the choice of k are given. Several examples are given. *G. E. Noether* (Boston, Mass.).

Johnson, N. L. Sequential procedures in certain component of variance problems. Ann. Math. Statist. **25**, 357–366 (1954).

The author compares three different sequential procedures for discriminating between two values of the ratio of variance components in a single one-way classification. Formulae for calculating the approximate ASN (average sample number) are derived. *R. P. Peterson* (Riverside, Calif.).

Duncan, David B. Multiple range and multiple F tests. Biometrics **11**, 1–42 (1955).

Let m_i $(i=1, \cdots, n)$ be normally (μ_i, σ^2) and independently distributed sample means. The paper deals with the much discussed question of how to pick significant differences m_i-m_j once some collective (F or range) test has indicated their presence. A useful survey of previous approaches is given. The rule recommended by the author goes as follows: Any m_i-m_j is significant if each and every subset of p means $(p=2, \cdots, n)$ that contains m_i, m_j is significantly heterogeneous according to the α_p-level studentized range test with $\alpha_p=1-(1-\alpha)^{p-1}$, α being the fundamental significance level. The motivation is in the spirit of classical test theory. *G. Elfving* (Helsingfors).

Anderson, Oskar. Ein exakter nicht-parametrischer Test der sogen. Null-Hypothese im Falle von Autokorrelation und Korrelation. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 130–143 (1954).

Ruist, Erik. Comparison of tests for non-parametric hypotheses. Ark. Mat. **3**, 133–163 (1955).

The first part of this paper is a good historical review. The remainder is largely concerned with considerations similar to those of Hoeffding [Proc. 2nd Symposium Math. Statist. Probability, 1950, Univ. of California Press, 1951, pp. 83–92; MR **13**, 479] and others, the main novelty being in the consideration of admissibility of the sign test (which is known to be minimax for many weight functions) for testing that the median of a density function is zero. For example, it is shown that, if the class of densities is suitably restricted, a slight modification of the sign test may be more powerful. *J. Kiefer* (Ithaca, N. Y.).

De Munter, P. Consistance de tests non-paramétriques pour la comparaison d'échantillons. Acad. Roy. Belg. Bull. Cl. Sci. (5) **40**, 1106–1119 (1954).

This paper contains material similar to that of that part of the author's thesis [Bruxelles, 1954; MR **16**, 499] concerned with an r-sample test. *J. Kiefer.*

De Munter, Paul. Comparaison d'un ou de plusieurs échantillons à un échantillon de référence. Bull. Inst. Agronom. et Stations Rech. Gembloux **22**, 224–231 (1954).

Contents similar to that of the paper reviewed above. *J. Kiefer* (Ithaca, N. Y.).

Hoel, Paul G. On a sequential test for the general linear hypothesis. Ann. Math. Statist. **26**, 136–139 (1955).

For the problem of the title, the method of weight functions employed by Wald [Sequential analysis, Wiley, New York, 1947; MR **8**, 593; see also Barnard, Biometrika **39**, 144–150 (1952); MR **14**, 65] is used to derive a class of sequential procedures which for a certain sequence of weight functions reduces to that suggested by Johnson [same Ann. **24**, 614–623 (1953); MR **15**, 638; see also Hoel, ibid. **22**, 483 (1951)]. *J. Kiefer* (Ithaca, N. Y.).

James, G. S. Tests of linear hypotheses in univariate and multivariate analysis when the ratios of the population variances are unknown. Biometrika **41**, 19–43 (1954).

The work of previous papers by the author [Biometrika **38**, 324–329 (1951); MR **13**, 762] and by Welch [ibid. **38**, 330–336 (1951); MR **13**, 762] on an extension of the Behrens-Fisher (BF) problem to the case of testing equality of several means is further generalized, as noted in the title. Applications, e.g., to two-factor experiments (with replications) and the multivariate generalization of the BF problem, are considered. The definition of the suggested test is too lengthy to be described here. *J. Kiefer.*

Fabián, Václav. Structural relation. Czechoslovak Math. J. **4**(**79**), 354–363 (1954). (Russian summary)

The author's summary states: "The author finds two conditions for a function C to be identifiable from a relation $\eta=C(\xi)$ between two random variables, and studies the dependence between the random variables if the observed quantities are obtained with random errors." The problem is inspired by the theorem of Reiersøl [Econometrica **18**, 375–389 (1950); MR **12**, 347] on the identifiability of a linear relation with both variables subject to normal error. [This theorem is also a consequence of a result of the reviewer's, Skand. Aktuarietidskrift **35**, 132–151 (1953); MR **14**, 776.] *J. Wolfowitz* (Ithaca, N. Y.).

Sarkar, D., and Laha, R. G. A modification of the variate-difference method. Econometrica **23**, 67–72 (1955).

An extension of the variate-difference method to cover the hypothesis that the series contains a number of harmonic terms of known frequency as well as a polynomial trend and an uncorrelated residual. *P. Whittle.*

Moriguti, Sigeiti. Optimality of orthogonal designs. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 75–98 (1954).

The paper considers designs for N observations which have the property that best unbiased linear (b.u.l.) estimators of some unknown parameters have minimum variance. As is customary, it is assumed that each observation is a linear combination of the parameters and an error, and

that the errors are independently distributed with mean zero and variance σ^2.

The discussion begins with a review of the weighing problem. It is shown that in weighing articles on an unbiased chemical balance the b.u.l. estimator of the weight of any particular article cannot have variance less than σ^2/N, and necessary and sufficient conditions are stated for attaining the minimum. This result was obtained by Hotelling [Ann. Math. Statist. 15, 297–306 (1944); MR 6, 93]. If the conditions hold simultaneously for all articles, the design matrix is an orthogonal matrix with elements 1 or -1, according as an article is put on the left-hand or right-hand scale.

For the case of the biased spring balance, it is shown that the minimum variance of a b.u.l. estimator of a weight is $4\sigma^2/N$, and necessary and sufficient conditions are given for attaining the minimum. If the minimum is attained simultaneously for all articles, the design is called "orthogonal". However, the design matrix D is not an orthogonal matrix, for its elements are 1 or 0, according as an article is or is not put on the scale, and $D'D$ has $N/2$ in the principal diagonal and $N/4$ elsewhere. This case has been discussed by Mood [ibid. 17, 432–446 (1946); MR 8, 478] and O. Kempthorne [The design and analysis of experiments, Wiley, New York, 1952; MR 13, 572].

The next subject is "multifactorial designs" as formulated by Plackett and Burman [Biometrika 33, 305–325 (1946); MR 8, 44]. There are r factors F_j $(j=1, \cdots, r)$ each with L levels, and the parameters to be estimated are the general mean β_0 and the sets $(\beta_{j1}, \cdots, \beta_{j,L-1})$ $(j=1, \cdots, r)$, which represent the main effects. It is proved that the maximum of the variance of the b.u.l. estimator of β_{jk} $(k=1, \cdots, L-1)$ for a particular j cannot be less than $L\sigma^2/N$, and necessary and sufficient conditions are stated for attaining the minimum. These conditions do in fact insure that the variance of the estimator of β_{jk} is $L\sigma^2/N$ for all k.

When the conditions hold for a particular j, the effect $(\beta_{j1}, \cdots, \beta_{j,L-1})$ is said to be "orthogonal" to the other effects. It is shown that in a multifactorial design the effect $(\beta_{j1}, \cdots, \beta_{j,L-1})$ is orthogonal to the other effects if and only if each level of F_j appears the same number of times; and together with any level of any other factor $F_{j'}$, each level of factor F_j appears the same number of times.

When the conditions hold for all j the design is said to be "orthogonal". It is proved that a multifactorial design is orthogonal if and only if for every j, each level of the factor F_j appears the same number of times (i.e., exactly N/L times); and for every j, j', each level of the factor F_j appears together with each level of the factor $F_{j'}$ the same number of times (i.e., exactly N/L^2 times).

Next considered are general factorial designs, in which all main effects are taken into account, and the following interactions: If an m-factor interaction of $F_{j_1}, \cdots, F_{j_m}$ is taken into account, then so is the interaction of any subgroup of $F_{j_1}, \cdots, F_{j_m}$. Results similar to those discussed above are obtained. *W. S. Connor* (New Brunswick, N. J.).

Theory of Games, Mathematical Economics

Bellman, Richard, and Gross, Oliver. Some combinatorial problems arising in the theory of multi-stage processes. J. Soc. Indust. Appl. Math. 2 (1954), 175–183 (1955).

The problem of finding the optimum order in which a given number n of dissimilar items should be successively processed through two machines so as to minimize the total idle time on the second machine has been brought by S. Johnson into the form: find the maximum of the expression

$$(1) \qquad \sum_{i=1}^{m} a_i - \sum_{i=1}^{m-1} b_i$$

as m varies from 1 to n, the a_i's and b_i's being the respective processing times. Johnson also solved the problem, and the authors present his solution, state that it extends to the case of three machines, and point out that it is quite difficult to extend it any further. In order to study the case of larger numbers of machines they resort to a continuous analogue, which in the case of two machines is equivalent to that of solving a zero-sum two-person game in which the minimizing player picks a function ϕ satisfying $0 \leqq \phi(t) \leqq 1$ and $\int_0^T \phi(t)dt = k$, the maximizing player picks a number u on $[0, T]$ and the payoff is $[\alpha \int_0^u \phi(t)dt + \beta]$. This game has trivial solutions which in the case of large n approximate to the solutions of the discrete problem. The authors state that the method extends without difficulty to the case of larger numbers of machines, and show explicitly how to extend it to three. *J. M. Danskin* (Washington, D. C.).

Gaddum, J. W., Hoffman, A. J., and Sokolowsky, D. On the solution of the Caterer Problem. Naval Res. Logist. Quart. 1 (1954), 223–229 (1955).

The authors develop an algorithm for solving a generalization of the problem proposed by W. Jacobs [same Quart. 1, 154–165 (1954); MR 16, 386], and give a numerical example of its use. *J. Kiefer* (Ithaca, N. Y.).

Blackwell, David. On multi-component attrition games. Naval Res. Logist. Quart. 1 (1954), 210–216 (1955).

The author considers a two-person zero-sum game with vector payoff in which the players begin with specified supplies of certain commodities. In an engagement of the game, player I chooses an integer u, $1 \leqq u \leqq r$, and player II an integer v, $1 \leqq v \leqq s$. There is an $r \times s$ matrix $Z = \|z(u, v)\|$ of vectors $z(u, v)$ specifying the resulting decreases in the various commodities. Some of the components of a given $z(u, v)$ refer to the commodity owned by player I the remainder to those of the other player. The game terminates when one of the players runs out of something. A strategy $f = \{f_m\}$ for I is a sequence of functions giving him at the mth stage a mixed strategy for the choice of u which depends on the outcome of the previous $m-1$ engagements. Analogously for the second player. The main theorem may be described as follows: Consider a certain zero-sum two-person scalar game gotten from Z by taking weighted averages of the components of each $z(u, v)$. If the value of this game is positive (i.e., the expectation for one play) then the first player is almost certain to win when both start with large supplies. Example. *T. L. Saaty* (Takoma Park, Md.).

Beckmann, Martin. A continuous model of transportation. Econometrica 20, 643–660 (1952).

The problem considered is the minimization of transportation cost in a continuous region. Given are the net production density $g(x, y)dxdy$, where x and y are the co-ordinates, the net export density $g(x, y)dxdy$ and the money cost of transportation $k(x, y, u)$ per unit of flow density $u(x, y)$. It is shown that a field $\phi(x, y)$ of transportation vectors is efficient if, among other conditions,

$$k(x, y, |\phi|)\frac{\phi}{|\phi|} = \text{grad } h(x, y),$$

where $h(x, y)$ is a potential function. As is the case for the usual discrete transportation models, the difference of potential for any pair of locations is not greater than the minimal transportation cost, and is equal to the latter if there is a positive flow between the two locations. These results are extended to situations with more than one commodity, for which it is pointed out that efficiency prices and market prices do not necessarily coincide (in the one-commodity case they do coincide).

H. S. Houthakker.

TOPOLOGY

***Nöbeling, Georg. Grundlagen der analytischen Topologie.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. x+221 pp. Broschiert, DM 33.00; ganzleinen, DM 36.60.

The emphasis in this textbook is on abstractness and generality. The point of view adopted is that a "topology" is a particular type of endomorphism of a partially ordered set (generalizing the traditional closure operator on the subsets of a space); further assumptions are made only as needed. Most of the theorems are in fact obtained for Boolean algebras, the classical theorems for spaces being deduced as special cases. The need for detailed consideration of the partial ordering means inevitably that some topological topics (for instance, Baire category and cut points) have to be omitted for lack of space, and others (such as points of condensation, identifications, and continua) treated rather sketchily; but on the whole the essential fundamentals of "point-set" topology have been fitted into this "pointless" setting. The proofs are full and clear, though briefly worded and with few concessions to heuristics. Deviations from the standard terminology occur (e.g., a "filter" may here contain the empty set, and "kompakt" means "countably compact, not necessarily T_2"), but are carefully noted. The book consists of three chapters and an appendix. Chapter 1 gives the preliminary material on partially ordered sets, homomorphisms, filters, and Zorn's lemma (deduced via Hausdorff's maximality principle, the axiom of choice being tacitly assumed). The chapter ends with a discussion of products of (not necessarily countably many) partially ordered sets; three types of product have to be considered.

The main results on topology are in a long second chapter, which begins with the basic properties of closed and open sets, neighborhoods, etc., for a "topology" satisfying: $A_1 \leqq A_2 \rightarrow \overline{A_1} \leqq \overline{A_2}$, $A \leqq \overline{A}$, and $\overline{\overline{A}} = \overline{A}$ (the last property being sometimes dispensed with). If the statements $\overline{0} = 0$ and $\overline{A_1 \vee A_2} = \overline{A_1} \vee \overline{A_2}$ are also true (they need not even be meaningful), the topology is "classical". The treatment of convergence includes both Moore-Smith and filter convergence in the notion of a "filtered function" $\{x_i\}$, for which a filter $\mathfrak{F}$ of subsets of the index set I is assigned; "almost all x_i" then means "all x_i for a set of i's in $\mathfrak{F}$". Later topics include separation axioms (Urysohn's lemma being formulated so as to apply to non-classical topologies); compactness, m-compactness and local compactness; connectedness and local connectedness; the behaviour of products (of algebras and spaces with classical topologies) with regard to all these properties; and imbedding and representation theorems, including the Wallman compactification for a T_1 topological Boolean algebra, and the Čech-Stone-Tychonoff compactification for a completely regular space with classical topology. Urysohn's metrization and imbedding theorems are deduced, but the metrizability of non-separable spaces is not considered. Some standard properties of arcs and Peano spaces are also obtained, and are among the few theorems not generalized to Boolean algebras.

Chapter 3 deals with uniform structures; broadly speaking, it is shown that the standard theory of uniform spaces can be carried over to Boolean algebras (with automatically classical topologies), and that every uniform Boolean algebra can be imbedded in a complete uniform space. The appendix derives those properties of infinite cardinals which are needed in the book, via the well-ordering theorem. [Except that all the alephs are printed upside down, the reviewer found very few misprints, none seriously misleading; however, on pp. 66, 67, some closure signs have been misplaced. On p. 108, line 3, the topology in $\mathfrak{W}$ is presumably not classical without further assumptions. The treatment of product spaces (pp. 142, 143) could have been somewhat simplified by the observation that $\mathfrak{E}_{i_0}$ is homeomorphic to a subset of $P_\gamma \mathfrak{E}_\gamma$.] *A. H. Stone* (Manchester).

Harary, Frank. The number of linear, directed, rooted, and connected graphs. Trans. Amer. Math. Soc. **78**, 445–463 (1955).

The paper is concerned with the solution of the problem of the number of nonisomorphic graphs of various kinds based on the counting polynomials. The author considers linear graphs, directed graphs, rooted and multiply rooted graphs, and two new generalizations, graphs of strength s and graphs of type t. (In a graph of strength s a pair of points are joined by not more than s lines; no distinction is made between the lines. In a graph of type t there are t different kinds of lines and a pair of points may be joined by up to t lines but no two of them may be of the same kind.)

Besides formulas for the number of nonisomorphic graphs with a given number of points and lines, and for the number of the various generalizations of graphs, formulas are also derived for the number of connected graphs of any type in terms of the total number of graphs of this type; and for the number of rooted connected graphs of any given type in terms of the number of unrooted connected graphs and the number of rooted graphs of the given type. The paper ends with a list of interesting counting problems which are still unsolved, among them the question of the number of nonisomorphic planar graphs with a given number of points and lines. *G. A. Dirac* (Vienna).

Smirnov, Yu. M. On the completeness of proximity spaces. Trudy Moskov. Mat. Obšč. **3**, 271–306 (1954). (Russian)

Detailed proofs are given of most of the results contained in a previous note by the author [Dokl. Akad. Nauk SSSR (N.S.) **88**, 761–764 (1953); MR **15**, 144]. There are some additional theorems, e.g. if a complete metrisable δ-space P is contained as a dense subset in a compact δ-space uP, $A \subset uP$ is a non-void closed G_δ-set, $A \cap P = 0$, then the potency of A is not less than the power of the continuum [this is, essentially, an extension of a result by E. Čech, Ann. of Math. (2) **38**, 823–844 (1937)]; the completion of a metrisable δ-space is the largest δ-extension of P satisfying the first countability axiom. There is also a purely topological

result: a completely regular space R is pseudocompact (i.e. every continuous real-valued function on R is bounded) if and only if every locally finite open covering contains a finite subcovering.

Reviewer's remark: the author's problem concerning validity of the above proposition without the complete regularity condition is answered in the negative, already for regular spaces, by a slight modification of spaces considered by E. Hewitt [Ann. of Math. (2) 47, 503–509 (1946); MR 8, 165] and J. Novák [Časopis Pěst. Mat. Fys. 73, 58–68 (1948); MR 10, 467]. *M. Katětov* (Prague).

Smirnov, Yu. M. On the dimension of proximity spaces. Dokl. Akad. Nauk SSSR (N.S.) 95, 717–720 (1954). (Russian)

Let P be a proximity space. The δ-dimension $\delta\mathrm{d}\,P$ is defined as the least n such that every finite δ-covering [cf. Yu. M. Smirnov, Mat. Sb. N.S. 31(73), 543–574 (1952); MR 14, 1107] of P has a refinement which is a δ-covering of order $\leqq n+1$; if no such n exists, then $\delta\mathrm{d}\,P=\infty$. A finite collection $\{\Gamma_i\}$ is called a δ-bordering of P if, for some compact $B\subset P$, $\{\Gamma_i\setminus B\}$ is a δ-covering of $P\setminus B$. The border δ-dimension $\delta\mathrm{d}^*\,P$ is defined as the least n such that there exist arbitrarily fine δ-borderings of order $\leqq n+1$; if no such n exists, then $\delta\mathrm{d}^*\,P=\infty$. The relative dimension $\mathrm{rd}\,R$ of a completely regular (c.r.) space R is defined [cf. P. S. Alexandroff, Proc. Roy. Soc. London. Ser. A. 189, 11–39 (1947); MR 9, 52] as $\sup_K \dim K$, $K\subset R$ being compact.

The relations between $\delta\mathrm{d}$, $\delta\mathrm{d}^*$, rd and dim are given by the following fundamental results:

$$\delta\mathrm{d}\,P=\dim uP,\quad \delta\mathrm{d}^*\,P=\delta\mathrm{d}(uP\setminus P),$$
$$\delta\mathrm{d}\,P=\max(\delta\mathrm{d}^*\,P,\ \mathrm{rd}\,P),$$

where uP denotes the compact δ-extension of P. As a corollary we have $\delta\mathrm{d}(\bigcup_{i=1}^{n}A_i)=\max \delta\mathrm{d}\,A_i$ (with a countable number of A_i, this equality is false, in general). The well known theorems concerning mappings in polytopes, extension of mappings into spheres and essential mappings are valid, with suitable modifications, for the δ-dimension. If A is a subset of the Euclidean space E^n, then $\delta\mathrm{d}\,A=n$ if and only if there exists $r>0$ such that every set $\{y\mid\rho(y,A)<\epsilon\}$, $\epsilon>0$, contains a spherical region of radius r (corollary: for bounded $A\subset E^n$, $\delta\mathrm{d}\,A<n$ if and only if A is nowhere dense).

In the last section of the note, the covering dimension $\dim R$ is defined for c.r. spaces by means of normal coverings [J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940; MR 2, 67; Yu. M. Smirnov, Mat. Sb. N.S. 31(73), 152–166 (1952); MR 14, 303]; the equalities $\dim R=\dim \beta R$, $\dim\bigcup_{i=1}^{n}A_i=\sup\dim A_i$ (A_i closed) are stated and the relations between the δ-dimension and the covering dimension of c.r. spaces are mentioned. For the dimension of c.r. spaces and the above equalities cf. also the reviewer's note [Časopis Pěst. Mat. Fys. 75, 79–87 (1950); MR 12, 119]. *M. Katětov* (Prague).

Lombardo-Radice, Lucio. La dimensione, invariante topologico. Archimede 7, 1–9 (1955).

Expository paper.

Gómez Aguilar, Ignacia. On Menger dimension in Kuratowski space. Gaceta Mat. (1) 6, 160–162 (1954). (Spanish)

The author gives a definition of dimension of closed sets in T_1 spaces which, she asserts, reduces to the usual one for Hausdorff spaces. *M. E. Shanks* (Lafayette, Ind.).

Iséki, Kiyoshi. On extension of continuous mappings on countably paracompact normal spaces. Proc. Japan Acad. 30, 736–740 (1954).

Some extension theorems for continuous functions are proved. [Reviewer's note: Unfortunately, the theorems in this paper are known. The assumption of countable paracompactness in Theorem 1 is superfluous, and the theorem without this assumption follows immediately from Dugundji, Pacific J. Math. 1, 353–367 (1951) [MR 13, 373] and Hanner, Ark. Mat. 1, 375–382 (1951) [MR 13, 266].] *E. Michael* (Seattle, Wash.).

Shimrat, M. Embedding in homogeneous spaces. Quart. J. Math., Oxford Ser. (2) 5, 304–311 (1954).

It is well known that each completely regular space S is a subset of some topological group and therefore of a space S^* with a transitive homeomorphism group. This latter type of embedding is here carried out without any separation axioms. It is shown that S^* has each of the following properties when S has them: T_0-T_4, complete regularity, connectedness, local connectedness, and others. When S is T_1, it is closed in S^*. A variant of the construction when S is metric imbeds S as a closed set isometrically into a metrically homogeneous space. *R. Arens.*

de Groot, J. On Cohen's topological characterization of sets of real numbers. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 33–35 (1955).

The author shows that a topological space M is homeomorphic to a subspace of the space of real numbers if and only if M is metrizable and separable, each point of M is of order at most two, and for each point x of order two, x is a cut point and M is locally connected at x. Earlier characterizations of such spaces have been given by L. W. Cohen [Fund. Math. 14, 281–303 (1929)] and by the author [Nederl. Akad. Wetensh. Proc. Ser. A. 50, 876–884 (1947); MR 9, 243]. *J. L. Kelley* (Berkeley, Calif.).

Salzmann, Helmut. Über den Zusammenhang in topologischen projektiven Ebenen. Math. Z. 61, 489–494 (1955).

The author proves some basic results concerning topological projective planes: the underlying topological space is regular; the plane and all its lines (projective or affine) are either all connected or all totally disconnected. In a note added in proof it is observed that there is some overlapping with a paper of Skornyakov [Trudy Moskov. Mat. Obšč. 3, 347–373 (1954); MR 16, 60]. *I. Kaplansky.*

Bing, R. H. Some monotone decompositions of a cube. Ann. of Math. (2) 61, 279–288 (1955).

Suppose that M' is a 2-sphere in 3-space, and that H' is an upper semi-continuous decomposition of M' into continua. A classical result of R. L. Moore states that the hyperspace (also denoted by H') is a cactoid. Suppose now that M is M' together with its interior, and that H is that decomposition of M whose elements are the elements of H' and points of the interior of M. What, then, is the hyperspace? A result of Floyd and Fort [Proc. Amer. Math. Soc. 4, 828–830 (1953); MR 15, 244] shows that if H' is a 2-sphere then H is a 2-sphere with interior. The author attacks the problem for a general N'. His striking first result is that if N' is a dendron, then N is a 3-sphere. The proof sets up a sequence of partitions of N which satisfy the hypotheses of his characterization of the 3-sphere in terms of partitions. For a general N with spherical cyclic elements $C_1, C_2, \cdots,$

for each i he adds to N an open 3-cell X_i such that X_i+C_i is a closed 3-cell. He then proves that the union of N and the X_i is a 3-sphere. Hence N is topologically equivalent to a cactoid C on the 3-sphere together with one of its complementary domains. As an open problem, it is pointed out that the embedding of C is not completely settled.

E. E. Floyd (Charlottesville, Va.).

Harrold, O. G., Jr. The enclosing of simple arcs and curves by polyhedra. Duke Math. J. **21**, 615–621 (1954).

A set K (in Euclidean 3-space E^3) is locally polyhedral at a point p if p has a closed neighborhood which meets K in either the empty set or a finite polyhedron. A set $J \subset E^3$ has property P if for each point x of J and each $\epsilon > 0$ there is a topological 2-sphere K such that (1) K lies in the ϵ-neighborhood of x, (2) x lies in the interior of K (that is, the bounded component of E^3-K), (3) the number of points in $K \cap J$ is the order of x in J, and (4) K is locally polyhedral at each point of $K-J$.

It is shown that property P is preserved under homeomorphisms of E^3 onto itself. Hence every tamely imbedded set J has property P, although the converse is false. Theorem 1. Let J be an arc (or simple closed curve) in E^3 having property P, and let ϵ be a positive number. Then there is a polyhedral 2-sphere K (or a polyhedral torus K) such that the interior of K contains J, and such that K lies in the ϵ-neighborhood of J. Theorem 2. Let J be an arc in E^3. If J has property P, then E^3-J (compactified at infinity) is an open 3-cell. But the converse is false. *E. E. Moise.*

Anderson, R. D., and Hamstrom, Mary-Elizabeth. A note on continuous collections of continuous curves filling up a continuous curve in the plane. Proc. Amer. Math. Soc. **5**, 748–752 (1954).

It is shown that if G is a continuous collection of non-degenerate continuous curves filling up a compact continuous curve M in the plane, then with respect to its elements as points G is either an arc or a simple closed curve. Alternatively, the theorem can be stated as follows: Let M be a locally connected (compact) continuum in the plane, and let f be a monotone interior mapping of M onto a metric space J. Suppose that for each point x of J, the set $f^{-1}(x)$ is non-degenerate and locally connected. Then J is either an arc or a simple closed curve. *E. E. Moise.*

Polak, A. I. On functions not assuming constant values on connected non-trivial sets. Dokl. Akad. Nauk SSSR (N.S.) **100**, 213–215 (1955). (Russian)

Suppose that $\{f_n\}$ is a sequence of strongly open maps from a locally compact connected metric space R to a locally connected metric space R', where $\{f_n\}$ converges uniformly to a light map f. It is proved that for each $y \in R'$ the sequence $\{f_n^{-1}(y)\}$ of sets converges in the topological sense to $f^{-1}(y)$. With the hypotheses on R and R' strengthened very slightly, this theorem is implicit in the work of G. T. Whyburn [Mem. Amer. Math. Soc., no. 1 (1950); MR **13**, 764] on approximately interior sequences. *E. E. Floyd.*

Williams, Robert F. A note on unstable homeomorphisms. Proc. Amer. Math. Soc. **6**, 308–309 (1955).

A homeomorphism f of a metric space X onto itself is said to be unstable if there exists a real number $\delta(f, X) > 0$ such that if $x, y \in X$ then there exists an integer $m(x, y)$ for which $\rho(f^m(x), f^m(y)) > \delta$. Examples of unstable homeomorphisms on compact or connected metric spaces are easily constructed. However, the reviewer [Proc. Amer. Math. Soc. **1**, 769–774 (1950); MR **12**, 344] in studying unstable homeomorphisms was unable to construct an example of an unstable homeomorphism on a compact, connected metric space. In this note three such examples are given. One example is on a solenoid, another is on a plane indecomposable continuum and, finally, by joining these spaces at a fixed point of their respective transformations a decomposable continuum is secured on which the induced homeomorphism is unstable. *W. R. Utz.*

Weier, Josef. Endliche Abbildungsscharen in kompakten topologischen Mannigfaltigkeiten. Monatsh. Math. **59**, 1–21 (1955).

The question considered is that of approximating a continuous family f^τ $(0 \leqq \tau \leqq 1)$ of mappings $P \to P$ by a second family each member of which leaves fixed at most a finite number of points. The following theorem is typical. Assume that P is a compact topological manifold. Then f^τ can be uniformly approximated by ϕ^τ such that the points (p, τ) satisfying $\phi^\tau p = p$, if there are any, constitute a finite number of arcs in $P \times I$ which are disjoint except for end-points.

P. A. Smith (New York, N. Y.).

Curtis, M. L. Classification spaces for a class of fiber spaces. Ann. of Math. (2) **60**, 304–316 (1954).

Let Q be some class of topological spaces (e.g., the class of finite polyhedra). The author defines a fibre space with respect to the class Q to be a triple (E, B, p), where E and B are topological spaces, p is a continuous map of E onto B, and the covering homotopy property holds for maps of any space $X \in Q$ into E. In the case where Q is the class of finite polyhedra, this definition is due to J. P. Serre [Ann. of Math. (2) **54**, 425–505 (1951); MR **13**, 574]. With this definition of fibre space, many of the basic concepts of the theory of fibre bundles [cf. parts I and II of N. E. Steenrod, The topology of fibre bundles, Princeton, 1951; MR **12**, 522] have more or less obvious analogues. In particular, two kinds of equivalence between fibre spaces are defined. One of them is essentially the "fibre homotopy equivalence" of R. Thom [Ann. Sci. Ecole Norm. Sup. (3) **69**, 109–182 (1952); MR **14**, 1004], while the other is a weaker kind of equivalence. The main purpose of this paper is to study to what extent the standard theorems on the classification of fibre bundles, existence of universal bundles, etc. [cf. Steenrod, loc. cit., §19] can be extended to fibre spaces with respect to some class Q. Most of the results are too involved to be stated here.

Reviewer's note: The proof given of Lemma 6.1 is incorrect: the map θ_1 needs to be continuous. The author informs me that he has a correct proof for the case where B is a locally finite complex which he intends to publish soon. Since this is the only case of this lemma actually needed in the sequel, the results of the rest of the paper are not invalidated by this error. It is apparently not known whether or not the lemma is true in the full generality that it is stated.

W. S. Massey (Providence, R. I.).

Toda, Hirosi. Sur les groupes d'homotopie des sphères. C. R. Acad. Sci. Paris **240**, 42–44 (1955).

In this note the author introduces certain homotopy constructions and some exact sequences involving the p-primary components of homotopy groups of spheres which he intends to use in a subsequent note [cf. the following review]. Only very brief indications of the necessary definitions, etc. are given. *W. S. Massey* (Providence, R. I.).

Toda, Hirosi. Calcul de groupes d'homotopie des sphères. C. R. Acad. Sci. Paris **240**, 147–149 (1955).

The main result of this note is contained in a table which gives the structure of the $(n+r)$-dimensional homotopy group of the n-sphere for $8<r<14$ and all n [the results for $r\leq 8$ were known previously; cf. J. P. Serre, same C. R. **236**, 2475–2477 (1953); MR **14**, 1110]. No proofs are given.

W. S. Massey (Providence, R. I.).

Chang, S. C. On the homotopy groups of $S^p \cup S^q$. I. Sci. Sinica **3**, 219–223 (1954).

Let X be a space which is the union of a p-sphere, S^p, and a q-sphere, S^q, having a single point in common. The two main results of this paper may be stated as follows: (a) If $p>q\geq 2$, then for $r\leq 2(p+q)-4$ the r-dimensional homotopy group $\pi_r(X)$ is isomorphic to the direct sum of the following groups: $\pi_r(S^p)$, $\pi_r(S^q)$, $\pi_{r+1}(S^{p+q})$, $\pi_{r+2}(S^{2p+q})$, and $\pi_{r+i}(S^{p+iq})$ for all $i>1$. (b) If $p=q$, $p\geq 2$, and $r\leq 4p-4$, then $\pi_r(X)$ is isomorphic to the direct sum of the following groups: $\pi_r(S^p)$, $\pi_r(S^p)$, $\pi_{r+1}(S^{2p})$, $\pi_{r+2}(S^{3p})$, and $\pi_{r+i}(S^{ip})$.

The main tools used in the proofs are a very general theorem of G. W. Whitehead on the homotopy groups of two spaces having a single point in common [Ann. of Math. (2) **51**, 192–237 (1950); MR **12**, 847] and a theorem of A. L. Blakers and the reviewer on triad homotopy groups [ibid. **55**, 192–201 (1952); MR **13**, 485].

W. S. Massey.

Chang, S. C. On the homotopy groups of $S^p \cup S^q$. II. Sci. Sinica **3**, 225–236 (1954).

The author continues his study of the homotopy groups $\pi_r(X)$ (same notation as the preceding review). In case (a), $p>q\geq 2$, he obtains partial information if $r=2(p+q)-3$. In case (b), $p=q$, $p\geq 2$, he obtains partial results if $r=4p-3$. His methods involve the technique introduced in 1937 by H. Freudenthal [Compositio Math. **5**, 299–314 (1937)].

Reviewer's note: These results on the homotopy groups of the union of two spheres are all special cases of the general results recently published by P. J. Hilton [see the paper reviewed below].

W. S. Massey (Providence, R. I.).

Hilton, P. J. On the homotopy groups of the union of spheres. J. London Math. Soc. **30**, 154–172 (1955).

The main purpose of this paper is to compute the homotopy groups of the union of spheres with a single point in common and also to give its applications; a proof of the Jacobi identity for triple Whitehead products and some results concerning the left distributive law of composite elements related to multiple Whitehead products. Let S_i be a sphere of dimension r_i+1, $r_i\geq 1$, $i=1, 2, \cdots, k$, and let T be the union of the spheres $S_1, \cdots, S_k$. Following P. Hall's definition of basic commutators with a minor modification, the author defines and orders the basic products among the positive generators of $\pi_{r_i+1}(S_i)$, $i=1, 2, \cdots, k$, by the Whitehead operation. Then a basic product of weight w is regarded as a string of symbols $l_{v_1}l_{v_2}\cdots l_{v_w}$, suitably bracketed, where $1\leq v_i\leq k$. If a basic product includes l_i w times, then the height of the basic product is called $\sum_{i=1}^{k} w_i r_i$. If the basic products are written and ordered as $p_1, p_2, \cdots$, whose heights are $q_1, q_2, \cdots$, then $\pi_n(T)\cong\sum_{i=1}^{\infty}\pi_n(S^{q_i+1})$, where any $\alpha\in\pi_n(S^{q_i+1})$ is embedded in $\pi_n(T)$ by the composition $p_i\circ\alpha$ with the basic product $p_i\in\pi_{q_i+1}(T)$. This theorem is essentially proved by using a theorem of Bott and Samelson [Comment. Math. Helv. **27**, 320–337 (1954); MR **15**, 643] and one of Samelson [Amer. J. Math. **75**, 744–752 (1953); MR **15**, 731]. Let Ω be the space of loops on T, then $H(\Omega)$ is the free associative ring, generated by $b_1=\rho(l_1), \cdots, b_k=\rho(l_k)$, where ρ: $\pi_{r_i+1}(S^{r_i+1})\to H_{r_i}(\Omega)$ is the natural composite homomorphism

$$\pi_{r_i+1}(S^{r_i+1})\to\pi_{r_i+1}(T)\to\pi_{r_i}(\Omega)\to H_{r_i}(\Omega).$$

For homogeneous elements $a\in H_p(\Omega)$, $b\in H_q(\Omega)$, define the quasicommutator of a and b by

$$a\circ b=(-1)^{pq}a\cdot b+(-1)^{p(q+1)}b\cdot a,$$

where $\cdot$ denotes the Pontrjagin multiplication in $H(\Omega)$. Using the $\circ$ operation instead of the Whitehead product operations, we can define the basic quasi-commutators just like the definition of basic products. Then, by known facts for commutators [Witt, J. Reine Angew. Math. **177**, 152–160 (1937)], the basic quasi-commutators monomials like $b_1^{n_1}b_2^{n_2}\cdots$ form a free additive base. Let Ω_i be the space of loops on S^{r_i+1}, then p_i: $S^{r_i+1}\to T$; representing the basic product p_i, induces the natural ring homomorphism f_i: $H(\Omega_i)\to H(\Omega)$. Next, consider the space $\Omega^*=\prod_{i=1}^{\infty}\Omega_i$ with some identifications; then we have a well defined additive isomorphism (not multiplicative) φ: $H(\Omega^*)\to H(\Omega)$. Then by an obvious extension of Whitehead's theorem [Serre, Ann. of Math. (2) **58**, 258–294 (1953); MR **15**, 548], we have the main theorem $\sum_{i=1}^{\infty}\pi_n(S^{r_i+1})\cong\pi_n(T)$. If we select T as $S_1\vee S_2\vee S_3$, then by definition $[l_1, [l_2, l_3]]$ is not the basic product, and is easily seen to be a linear combination of $[l_2, [l_1, l_3]]$ and $[l_3, [l_1, l_2]]$. Considering the homomorphism ρ and applying Samelson's theorem, the Jacobi identity is proved. This theorem is extended to the relative case. Let Φ: $\pi_n(S^r)\to\pi_n(S_1^r\vee S_2^r)$ be the homomorphism induced by pinching the equator of S^r, and let H'_{i-1} $(i=1, 2, \cdots)$ project $\pi_n(S_1^r\vee S_2^r)$ on its $(i+2)$nd direct summand in the decomposition given by the main theorem; then we have homomorphisms

$$H_0'\circ\Phi=H_0\colon\ \pi_n(S^r)\to\pi_n(S^{2r-1}),\quad H_1'\circ\Phi=H_1,$$
$$H_2'\circ\Phi=H_2\colon\ \pi_n(S^r)\to\pi_n(S^{2r-2}),\ \cdots,$$
$$\Phi\gamma=(l_1+l_2)\circ\gamma=l_1\circ\gamma+l_2\circ\gamma+[l_1, l_2]\circ H_0\gamma$$
$$+[l_1, [l_1, l_2]]\circ H_1\gamma+[l_2, [l_1, l_2]]\circ H_2\gamma+\cdots.$$

The author also proves theorems about these homotopy elements by the full use of properties of multiple Whitehead products. Among others, some of these are stated as follows. Let $\alpha\in\pi_n(S^r)$, $\beta\in\pi_r(X)$, and let k be an integer. Then the expansions of $k\beta\circ\alpha$ are given. Let $\alpha\in\pi_l(S^r)$, $\beta\in\pi_m(S^r)$, $\gamma\in\pi_n(S^r)$; then $[\alpha, [\beta, \gamma]]=0$ if r odd, $3[\alpha, [\beta, \gamma]]=0$ if r even, and all q-tuples products in S^r vanish for $q>3$. Finally, the question of the homomorphisms H_{i-1} is raised, and the author promises to come back to this in a later paper.

H. Uehara (Fukuoka).

Cartan, Henri. Sur l'itération des opérations de Steenrod. Comment. Math. Helv. **29**, 40–58 (1955).

In this paper the author determines explicitly the relations existing among Steenrod iterated operations.

Let $\mathcal{P}^k$: $H^q(X; Z_p)\to H^{q+2k(p-1)}(X; Z_p)$ be the Steenrod operations [Proc. Nat. Acad. Sci. U. S. A. **39**, 217–223 (1953); MR **14**, 1006]. For a an integer ≥ 0, congruent to 0 or 1 mod $2p-2$, define St_p^a: $H^q(X; Z_p)\to H^{q+a}(X; Z_p)$ as follows: for $p=2$, $\mathrm{St}_2^a=\mathrm{Sq}^a$ is the squaring operation; for p an odd prime and $a=2k(p-1)$, $\mathrm{St}_p^a=\mathcal{P}^k$, and, for $a=2k(p-1)+1$, $\mathrm{St}_p^a=\beta\circ\mathcal{P}^k$, where

$$\beta\colon\ H^q(X; Z_p)\to H^{q+1}(X; Z_p)$$

is the Bockstein-Whitney homomorphism. For each sequence $I=(a_1, \cdots, a_k)$ of integers $a_i\geq 0$, congruent to 0 or 1 mod $2p-2$, consider the operation $\mathrm{St}_p^I=\mathrm{St}_p^{a_1}\circ\cdots\circ\mathrm{St}_p^{a_k}$.

The degree of I is $\sum a_i$ and St_p^I is an endomorphism of degree q of $H^*(X; Z_p)$.

Let M_p be the algebra over Z_p with base the set of finite sequences I of integers $\geqq 0$ congruent to 0 or 1 mod $2p-2$. For every space X, the map $I \to \mathrm{St}_p^I$ defines a homomorphism of the graded algebra M_p onto a graded subalgebra of the algebra of endomorphisms of $H^*(X; Z_p)$. Let $R_p(X)$ be the kernel of this homomorphism and let R_p be the intersection of the $R_p(X)$ relative to all possible spaces X. The algebra $A_p = M_p/R_p$ is defined by the author as the "Steenrod algebra" relative to p. For all X, $H^*(X; Z_p)$ is a graded left module over the graded algebra A_p. The two-sided ideal R_p is a direct sum of the components R_p^q for the various degrees q. The elements of R_p^q are linear combinations of St_p^I of degree q operating as zero for every space X (i.e., relations on iterated Steenrod operations). For such a linear combination to be zero as operation for all X, it is necessary and sufficient that it be zero when it is applied to the fundamental class of an Eilenberg-MacLane space $K(Z_p, n)$ with n large enough.

A sequence $I = (a_1, \cdots, a_k)$ as considered above is called admissible if $a_i \geqq pa_{i+1}$. Using the explicit determination of the mod p cohomology of Eilenberg-MacLane groups $H(Z_p, n)$, due to the author for prime $p \geqq 2$ [ibid. **40**, 467–471, 704–707 (1954); MR **16**, 390] and Serre for $p=2$ [Comment. Math. Helv. **27**, 198–232 (1953); MR **15**, 643], the following fundamental theorem is obtained. The St_p^I relative to admissible sequences I form a base for the Steenrod algebra A_p. Consequently, any St_p^J can be expressed as a linear combination of elements St_p^I with I admissible. To do this, it is enough to express in terms of the base, for $p=2$, $\mathrm{Sq}^k \circ \mathrm{Sq}^h$ with $k<2h$, and for $p>2$, $\mathcal{P}^k \circ \mathcal{P}^h$ with $k<ph$ and $\mathcal{P}^k \circ \beta \circ \mathcal{P}^h$ with $k \leqq ph$. We have

$$(1)\quad \mathrm{Sq}^k \circ \mathrm{Sq}^h = \sum_i c^i_{k,h} \mathrm{Sq}^{h+k-i} \circ \mathrm{Sq}^i,$$

$$(2)\quad \mathcal{P}^k \circ \mathcal{P}^h = \sum_i c^i_{k,h} \mathcal{P}^{h+k-i} \circ \mathcal{P}^i,$$

$$(3)\quad \mathcal{P}^k \circ \beta \circ \mathcal{P}^h = \sum_i a^i_{k,h} \mathcal{P}^{h+k-i} \circ \beta \circ \mathcal{P}^i + \sum_i b^i_{k,h} \beta \circ \mathcal{P}^{h+k-i} \circ \mathcal{P}^i,$$

where the coefficients are explicitly given mod p binomial coefficients.

The method used by the author to obtain the relations is an extension of a method used by Serre [loc. cit.] and Thom [ibid. **28**, 17–86 (1954); MR **15**, 890]. When one knows a base of A_p the relations are established in a "universal space" consisting of a product of lens spaces $K(Z_p, 1)$.

The author's work makes explicit the Steenrod algebra A_p and it completely determines the Steenrod operations in $H^*(Z_p, n, Z_p)$.

Remark: Formulae (1) were first established by the reviewer [Thesis, Princeton, 1952; Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952); MR **14**, 306] and also by Serre [loc. cit.] and Thom [loc. cit.] Formulae (2) and (3) were obtained independently, by different methods, by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **39**, 636–638 (1953); MR **15**, 53] and the author. *J. Adem* (Mexico, D. F.).

GEOMETRY

Villa, Mario. L'indipendenza del postulato di Euclide. Archimede **7**, 22–29 (1955).

Gambier, M.-B. Trisectrices des angles d'un triangle. Ann. Sci. Ecole Norm. Sup. (3) **71**, 191–212 (1954).

Etude géométrique sur le théorème de Morley; des points d'intersection des trisectrices d'un triangle arbitraire on peut former 27 triangles équilatéraux. *O. Bottema* (Delft).

Lubin, Clarence. A proof of Morley's theorem. Amer. Math. Monthly **62**, 110–112 (1955).

Morley's theorem: the points of intersection of the adjacent trisectors of a triangle are the vertices of an equilateral triangle, is proved here by means of complex numbers. *O. Bottema* (Delft).

Barlotti, Adriano. Una proprietà degli n-agoni che si ottengono trasformando in una affinità un n-agono regolare. Boll. Un. Mat. Ital. (3) **10**, 96–98 (1955).

Di Noi, Salvatore. Geometria euclidea sulla sfera. Archimede **7**, 10–14 (1955).

Thébault, Victor. A propos du tranchet d'Archimède. Enseignement Math. **40** (1951–1954), 62–69 (1955).

Šubnikov, A. V. Simmetriya i antisimmetriya konečnyh figur. [Symmetry and antisymmetry of finite figures.] Izdat. Akad. Nauk SSSR, Moscow, 1951. 172 pp. 8 rubles.

Die gewöhnliche Symmetrie kann als regelmässige Anordnung gleicher Figuren untereinander definiert werden. Dabei können die Figuren entweder kongruent oder enantiomorph (=spiegelbildlich kongruent) sein. Die Symmetrieoperationen bilden eine Gruppe. Für Systeme mit mindestens einem Fixpunkt sind diese Operationen Drehungen oder Drehinversionen. Diese klassische Symmetrielehre kann durch Einführung des Vorzeichens der Figuren zu einer Lehre der Antisymmetrien erweitert werden. Antigleich sollen zwei Figuren dann heissen, wenn sie entweder kongruent oder enantiomorph sind, aber verschiedenes Vorzeichen (oder Farbe z.B.) aufweisen. (Anschauliches Beispiel: ein linker, bzw. rechter Handschuh, je entweder innen schwarz aussen weiss oder aussen schwarz innen weiss; 4 Möglichkeiten). Für "kristallographische" Systeme gibt es die folgenden Symmetrieoperationen: Drehachsen 1, 2, 3, 4, 6, Drehinversionsachsen $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{6}$, Antidrehachsen $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$, $\underline{6}$, und Antidrehinversionsachsen $\underline{\bar{1}}$, $\underline{\bar{2}}$, $\underline{\bar{3}}$, $\underline{\bar{4}}$, $\underline{\bar{6}}$. Die klassischen Symmetrietransformationen dreidimensionaler Figuren können auch als Operationen in einem vierdimensionalen Raum gedeutet werden, wobei die x_4-Achse, längs der die Koordinaten aller Punkte der Figuren null sind, am Platz bleibt; in tensorieller Darstellung

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

Die neuen Antisymmetrieoperationen hingegen werden durch den Tensor

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

dargestellt. Der Vorzeichenwechsel $+0 \to -0$ der x_4-Koordinaten entspricht der Antisymmetrie. Es gibt 58 kristallographische Antisymmetriegruppen endlicher Figuren [und

1151 analoge Raumgruppen; vgl. A. V. Šubnikov, Trudy Inst. Krist. 10, 10–12 (1954)]. *W. Nowacki* (Bern).

Neville, E. H. Notes on conics. 18: The theorems of Laguerre. Math. Gaz. 39, 37–41 (1955).

Laguerre has given in 1865 a series of theorems on the lengths of tangents and normals from a point to an algebraic curve. E.g., if $f(x, y)=0$ is the curve, $g(x, y)=0$ the aggregate of isotropic tangents, the product of the squares of the normals from (p, q) is a constant multiple of $f^2(p, q)g(p, q)$. The theorems are applied to conics. *O. Bottema.*

Hohenberg, Fritz. Ein einfacher Beweis des Satzes von Pohlke. Elem. Math. 10, 40–42 (1955).

Kadeřávek, František. A simple proof of a theorem related to a theorem of Pelz. Časopis Pěst. Mat. 79, 249–251 (1954). (Czech)

Wunderlich, W. Über die Evolutoiden der Ellipse. Elem. Math. 10, 37–40 (1955).

Sydler, J.-P. Quelques propriétés de la configuration complémentaire de Desargues. Elem. Math. 10, 32–37 (1955).

From the Desargues configuration $(10_3, 10_3)$, the author derives a "complementary" $(15_2, 10_3)$ by using the same ten lines and their remaining fifteen points of intersection, so that there are three points on each line but only two lines through each point. He observes that this configuration contains k-gons where $k=5$, 6, 8 or 9. For such a polygon $A_1A_2\cdots A_k$, he defines A_{ij} to be the remaining point on the side A_iA_j, and B_{kij} to be the point of intersection $A_kA_j\cdot A_{ki}A_{ij}$. The following results are typical. When $k=5$, the three points B_{123}, B_{345}, A_{51} are collinear (and likewise B_{234}, B_{451}, A_{12}; etc.). When $k=6$, the three points B_{123}, B_{345}, B_{561} are collinear (and likewise B_{234}, B_{456}, B_{612}). When $k=8$, the three lines $B_{123}B_{345}$, $B_{567}B_{781}$, A_1A_5 are concurrent. When $k=9$, the three points $B_{123}B_{345}\cdot A_1A_5$, $B_{567}B_{789}\cdot A_5A_9$, A_{19} are collinear. *H. S. M. Coxeter* (Toronto, Ont.).

Poli, L. Géométries finies. Rev. Questions Sci. (5) 16, 161–181 (1955).

An elementary expository paper. *L. J. Paige.*

Longo, Carmelo. Fasci di complessi lineari di rette in S_h. Boll. Un. Mat. Ital. (3) 9, 381–385 (1954).

A projective space S_h of dimension h is called a total space of a linear line complex if every line of S_h belongs to the complex. The general total spaces of a linear line complex in S_{2r+1} with maximum dimension are S_r depending on $(r+1)(r+3)/2$ parameters, and any two such spaces S_r do not intersect. By using these properties, a canonical form of the equation of a linear line complex in S_{2r+1} and a classification of the pencils of linear line complexes in S_5 are obtained. *C. C. Hsiung* (Bethlehem, Pa.).

Convex Domains, Extremal Problems, Integral Geometry

Steinhaus, H. Quelques applications des principes topologiques à la géométrie des corps convexes. Fund. Math. 41, 284–290 (1955).

The author defines a convex body V to be a domain of space bounded by a closed surface S such that for each point R of S there exists a unique plane $\pi(R)$ which has only R in common with S. By utilizing some well-known topological results (e.g. the Borsuk-Ulam Theorem, and the theorem on the non-existence of a continuous tangent vector field on a sphere), a number of properties of convex bodies are established. Among these are the following: 1) Through each interior point P of V there is a (plane) section of V having P as its centroid. 2) The set of sections of V whose centroid is also the centroid of another section has the power of the continuum. 3) The set of lines through each of which two distinct planes which cut V into equal volumes pass, has the power of the continuum. 4) The set of planes each of which divides both V and S into equal parts has the power of the continuum. 5) There is a plane whose intersections with V and S have the same centroid and which divides V (or S) into equal parts. 6) Given the additional property that at each point of S the principal radii of curvature exist and their lengths are continuous functions of S, then the points of S can be divided into two categories: The first consisting of pairs of points having congruent neighborhoods, and the second of limits of such pairs.

Haskell Cohen (Baton Rouge, La.).

Belgodère, Paul. Sur les surfaces minima en géométrie de Minkowski. C. R. Acad. Sci. Paris 240, 1504–1505 (1955).

The reviewer has shown [Comment. Math. Helv. 24, 156–187 (1950); MR 12, 527] that a hyperplane region P homeomorphic to a solid $(n-1)$-sphere bounded by a surface B homeomorphic to S_{n-2} in an n-dimensional Minkowski space minimizes the $(n-1)$-dimensional Minkowski area among all hypersurfaces homeomorphic to P and bounded by B. The author extends this result (with proofs barely indicated) to sufficiently small pieces P of minimal surfaces, i.e. surfaces for which the first variation of the $(n-1)$-dimensional Minkowski area vanishes.

H. Busemann (Los Angeles, Calif.).

Hsiung, Chuan-Chih. Some integral formulas for closed hypersurfaces. Math. Scand. 2, 286–294 (1954).

Let V_n be an orientable, twice differentiable hypersurface imbedded in a Euclidean space E_{n+1} $(n+1\geq 3)$ and let M_r be the rth mean curvature of V_n. The author proves the integral formula

$$\int_{V_n} M_{r+1}p\,dA+\int_{V_n} M_r\,dA=0 \quad (r=0, 1, \cdots, n-1),$$

where $p=p(P)$ denotes the "oriented" distance from a fixed point O to the tangent hyperplane at P and dA is the area element of V_n at P. Consequences of this formula are the following theorems. a) Suppose that there exist a point O and an integer s, $1\leq s\leq n-1$, such that $M_s>0$ and either $p\leq -M_{s-1}/M_s$ or $p\geq -M_{s-1}/M_s$ at all points of V_n; then V_n is a hypersphere. b) Suppose that there exist a point O and an integer s, $1\leq s\leq n$, such that at all points of V_n the function p is of the same sign, $M_i>0$ for $i=1, \cdots, s$ and M_s is constant; then V_n is a hypersphere. Some particular cases of these theorems (referring either to convex hypersurfaces or to the case $n=2$) are well known and the author gives the corresponding references. *L. A. Santaló.*

Łukaszewicz, J., and Steinhaus, H. On determining the "centre of copper" of a telephone network. Zastos. Mat. 1, 299–307 (1 plate) (1954). (Polish. Russian and English summaries)

The authors present various methods of finding the minimum of $\sum c_i d(P, P_i)$, where the c_i are positive constants,

P a variable point and the P_i given fixed points in the plane, and $d(P, Q)$ denotes Euclidean distance.

J. M. Danskin (Washington, D. C.).

Algebraic Geometry

Rosina, B. A. Sul numero dei diametri autoconiugati e sul numero delle coppie di diametri mutuamente coniugati di una curva algebrica piana con punti all'infinito non tutti distinti. Ann. Univ. Ferrara. Sez. VII. (N.S.) **3**, 67–74 (1954).

A diameter of a plane curve is the polar line of a point at infinity. Two diameters are conjugate if each is the polar of the point at infinity on the other, and a diameter is self-conjugate of its own point at infinity. It is shewn by very elementary calculations that if a curve has h distinct points at infinity (regardless of their multiplicity or the order of the curve) it has at most n distinct proper self-conjugate diameters, and likewise at most $\binom{n-1}{2}$ pairs of conjugate diameters, unless it has an infinity of such pairs. In particular, if $h=1$ there are no proper conjugate pairs; if $h=2$ there are none or an infinity according as the two points count unequally or equally among the intersections of the line at infinity with the curve. *P. Du Val* (London).

Godeaux, Lucien. Sur l'existence de surfaces multiples possédant des points de diramation de structure donnée. Rend. Mat. e Appl. (5) **14**, 42–47 (1954).

First the author summarises, perhaps more clearly and succinctly than he has done elsewhere, the complicated theorem which he has built up cumulatively in a large number of previous papers [cf. MR **14**, 578, 897; **15**, 645]. This theorem is: Let F be a surface on which is a cyclic involution I of prime order $p>2$, having only a finite number of isolated united points, and Φ the projective model of a linear system compounded with the involution, such that each united point corresponds to a single point of Φ; the neighbourhood on F of any united point either consists wholely of united points, or contains an involution subordinated to I; in the former case the corresponding point A of Φ is p-ple; in the latter it depends on a particular pair of residues $\alpha, \beta \pmod p$ satisfying $\alpha\beta\equiv 1 \pmod p$, as follows: if λ_1, μ_1 is that one of the $\frac{1}{2}(p+3)$ solutions of the equivalent congruences

$$\lambda+\alpha\mu\equiv 0, \quad \beta\lambda+\mu\equiv 0 \pmod p$$

for which $\lambda+\mu$ is least, and if $h_1, h_2, a_1, b_1, a_2, b_2, m, n$, are defined by the relations

$$\lambda_1+\alpha\mu_1=h_1p, \quad \beta\lambda_1+\mu_1=h_2p,$$
$$p=a_1\alpha_1+b_1=a_2+b_2\beta \quad (b_1<\alpha, a_2<\beta)$$
$$m\alpha<h_1b_1<(m+1)\alpha, \quad n\beta<h_2a_2<(n+1)\beta$$

the point A of Φ is of multiplicity $a_1+m+n+b_2$, the tangent cone having four rational constituents of order a_1, m, n, b_2, of which the first and second, the second and third, and the third and fourth have each a common generator; there may also be double points, either conic nodes or binodes, in the neighbourhood of A along these common generators. (As m or n or both may be zero, the cone may in fact have as few as two constituents.)

The present paper is devoted to shewing that such a point can in fact exist for the most general values of the constants defined above, by actually constructing a fairly simple example. *P. Du Val* (London).

Godeaux, Lucien. Sur la structure de certains points de diramation d'une surface multiple d'ordre 37. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 329–342 (1955).

Gallarati, Dionisio. Sopra una superficie dell'ottavo ordine dotata di 157 nodi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 454–459 (1954).

Extending methods previously used by himself [Atti Accad. Ligure **9**, 106–112 (1953); MR **15**, 250] and B. Segre [ibid. **9**, 15–22 (1953); MR **15**, 249], the author constructs a surface in S_3 of order eight with 157 nodes.

R. J. Walker (Los Angeles, Calif.).

Gallarati, Dionisio. Ricerche sugli spazi lineari di una ipersuperficie algebrica. Atti Accad. Ligure **10** (1953), 87–96 (1954).

The author finds that if $r-1$ primals $V_{r-1}^{m_i}$, of degree m_i, of a projective space S_r of dimension r have a common linear k-dimensional subvariety S_k which has the multiplicity s_i for $V_{r-1}^{m_i}$, and if $V_{k-1}^{d_i}$ is the subvariety of S_k of multiplicity s_i+1 for $V_{r-1}^{m_i}$, then generally there are $\sum\prod(m_i-s_i-d_i)\prod s_j$ points of S_k in which the $r-1$ primals $V_{r-1}^{m_i}$ have a common tangent $(k+1)$-dimensional subspace, where $\prod(m_i-s_i-d_i)$ is the product of k factors, $\prod s_j$ is the product of the $r-1-k$ factors whose indexes don't figure in the above product, and the sum must be extended to all the combinations of order k among the numbers $1, \cdots, r-1$ corresponding to the indexes of the first product. To obtain this result, the author employs models $\bar{V}^i_{r-2}$ without singular points, contained in the Segre variety $S_k\times S_{r-k-1}$, which are the images of the pairs (P, S^i_{k+1}), where P is a point of S_k and S^i_{k+1} is a generator subspace of the tangent hypercone to $V_{r-1}^{m_i}$ in P. He expresses $\bar{V}^i_{r-2}$, by means of a linear equivalence, in the form $\bar{V}^i_{r-2}\equiv(m_i-s_i-d_i)A^i+s_iB^i$, where $A^i=S'_{k-1}\times S''_{r-k-1}$, $B^i=S_k'\times S''_{r-k-2}$ and $S'_{k-1}\subset S_k'$, $S''_{r-k-2}\subset S''_{r-k-1}$. This result is obtained by determining the number of common points to all the $\bar{V}^i_{r-2}$ $(i=1, \cdots, r-1)$. *P. Abellanas.*

Chatelet, François. Exemples de surfaces de Brauer. C. R. Acad. Sci. Paris **239**, 1578–1579 (1954).

In an earlier paper [Ann. Sci. Ecole Norm. Sup. (3) **61**, 249–300 (1944); MR **7**, 323] the author applied some general algebraic techniques to obtain properties of algebraic varieties defined by polynomials with coefficients in a given field. Later [C. R. Acad. Sci. Paris **236**, 268–269 (1953); MR **14**, 790] he applied these methods to cubic surfaces containing a rational homoloidal net of skew cubic curves. In the present note he outlines a method of constructing all such surfaces having three conical nodes, gives an example, and comments that the method can be generalized to higher-dimensional varieties. *R. J. Walker.*

Terracini, Alessandro. Una classe di superficie razionali iperspaziali con asintotiche razionali normali. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 263–270 (1954).

Cet article traite d'une classe de surfaces rationnelles hyperspatiales admettant un système d'asymptotiques constitué par des courbes rationnelles normales. Ces surfaces sont celles de l'espèce indiquée, si l'on impose aux courbes rationnelles normales mentionnées la condition d'appartenir à certains systèmes spéciaux définis et déterminés par l'auteur. Dans l'espace projectif S_3 les surfaces envisagées se réduisent à la surface réglée de Cayley. Pour S_r $(r>3)$ les surfaces en question cessent d'être réglées, mais elles possèdent un systéme remarquable de ∞^1 courbes, se réduisant au système des ∞^1 génératrices rectilignes pour $r=3$, et à

partir desquelles on peut obtenir une construction de ces surfaces généralisant la construction bien connue de la surface de Cayley. *P. Vincensini* (Marseille).

Segre, Beniamino. Sui sistemi di forme quadratiche nel campo reale. Comment. Math. Helv. **28**, 288–300 (1954).

Let Σ_δ $(0 \leqq \delta \leqq \frac{1}{2}n(n+3)-1)$ denote a δ-fold linear system of real quadrics (given by equations with real coefficients) in a real projective space S_n $(n \geqq 1)$; and let $\rho(\Sigma_\delta)$ be the dimension of the largest real linear space S_ρ that lies on any quadric of Σ_δ. The following problems are propounded: (I) to find, for given n and δ, the lower bound $\rho = \rho(n, \delta)$ of $\rho(\Sigma_\delta)$ as Σ_δ varies; (II) to find, for given n and ρ, the least $\delta = \delta(n, \rho)$ such that $\rho(\Sigma_\delta) \geqq \rho$ for all Σ_δ. A complete solution of (I) would lead to a solution of (II). The case $\rho = n-1$ of (II) reduces (by duality) to the following: (III) to find the largest $d = d(n)$ such that for every Σ_d there exists a pair of real points which are conjugate for every quadric of Σ_d. This last problem is substantially that previously propounded by H. Hopf [Vierteljschr. Naturf. Ges. Zürich **85** Beiblatt, 165–177 (1940); MR **2**, 321] and solved by him for the values 1, 2, 3 of n.

The methods of the present paper are algebro-geometrical and the central result is the identification of an infinite class of pairs of integers n, ρ such that

$$\delta(n, \rho) = \tfrac{1}{2}(2\rho-n+1)(2\rho-n+2).$$

This last expression is in any case a lower bound for $\delta(n, \rho)$ since the generic Σ_δ $(\delta \geqq \delta(n, \rho))$ must contain at least one $(2\rho-n+1)$-fold specialized quadric with real or complex generators S_ρ. If we write $\delta_0 = \frac{1}{2}(2\rho-n+1)(2\rho-n+2)$, and if the real quadrics of S_n are represented by the points of a real S_ν $(\nu = \frac{1}{2}n(n+3))$, then any $\sum_{\delta_0}$ is represented in S_ν by a space S_{δ_0}, and the $(2\rho-n+1)$-fold specialized quadrics of S_n are represented by points of an irreducible manifold Φ of dimension $\nu-\delta_0$. The type of solution investigated here is that which arises when every S_{δ_0} of S_ν meets Φ in at least one point which either (i) represents a quadric of S_n, exactly $(2\rho-n+1)$-fold specialized, with real generating S_ρ, or (ii) represents a more highly specialized quadric of S_n which contains at least one real S_ρ. The result obtained is that, for $[n/2] \leqq \rho < n$, a sufficient condition for $\delta(n, \rho)$ to be equal to δ_0 is that the order of Φ (given by a known formula) should be odd. [The case $\rho = \frac{1}{2}(n-1)$, when Φ does not exist, is excluded.] The basic continuity arguments used in proving this are stated as follows: (a) if a real V_δ, varying continuously in a real S_ν, contains always some real S_ρ, then every limiting position of V_δ has the same property; (b) a real V_δ of odd order meets every $S_{\nu-\delta}$ in some real point; (c) if W is a δ-dimensional portion (locus of real simple points) of an algebraic V_δ of S_ν, such that the boundary of W belongs to an algebraic sub-variety U of V_δ, and if a real $S_{\nu-\delta}$, varying continuously without ever meeting U, meets W always in at most a finite number of points, then this number is either always even or always odd; (d) an algebraic V_δ, with real points, non-singular, of dimension $\delta \geqq 2$, and linearly connected, remains connected when there are removed from it the points of any algebraic sub-variety of dimension $\leqq \delta-2$.

In the final section of the paper, the author investigates arithmetically the condition that the order of Φ should be odd, and he obtains particular forms for n and ρ such that $\delta(n, \rho) = \delta_0$. One very special case is given by the formulae $n = 2^h$, $\rho = 2^h-1$, where h is any positive integer; and this gives an infinite class of solutions $\alpha(n) = 2^{h+1}-1$ of problem (III). Among other special results we quote one which says that, in a real space S_{2m} every pencil of real quadrics always contains a (specialized) quadric on which there lies a real S_m.

J. G. Semple (London).

Segre, Beniamino. Alcune questioni di realtà. Univ. e Politecnico Torino. Rend. Sem. Mat. **13**, 31–32 (1954).

Summary of the paper reviewed above.

Genovese, Carla. Una classe di superficie W dello spazio a cinque dimensioni. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 243–249 (1954).

L'auteur considère, dans l'espace projectif S_5, une surface algébrique W comme représentée paramétriquement par des équations de la forme $x^i = k_i u_1^{a_i} u_2^{b_i}$ $(i = 1, 2, \cdots, 5)$, et il effectue la recherche de celles de ces surfaces représentant une équation de Laplace, pour lesquelles il existe une polarité nulle transformant chaque point en l'hyperplan osculateur correspondant. Il montre que les surfaces répondant à la question (l'équation de Laplace étant du type non parabolique) sont celles dont la représentation paramétrique peut être mise sous la forme

$$x^{(1)} : x^{(2)} : x^{(3)} : x^{(4)} : x^{(5)} : x^{(6)} = 1 : s : t : s^\alpha t^\beta : s^{1-\alpha} : t^{1-\beta} : st.$$

P. Vincensini (Marseille).

Gherardelli, Francesco. Covarianti birazionali di sistemi lineari di curve sopra una superficie algebrica. Ann. Mat. Pura Appl. (4) **37**, 157–174 (1954).

If Φ is a surface in projective space S_r, an osculating space of order k, $S(k)$, in general is of dimension $\frac{1}{2}(k+1)(k+2)-1$. Define ρ and s by $r = \frac{1}{2}(\rho+1)(\rho+2)-1+s$, $0 \leqq s \leqq \rho+1$. If $s > 0$, a hyperplane which contains the $S(\rho)$ osculating Φ at a point P intersects Φ in a curve having P as a $(k+1)$-fold point. For certain such hyperplanes two of the $k+1$ tangents to the curve at this point coincide. These define the principal directions at the point; in general they are distinct. Let $J_{\rho+1,s}$ be the locus of points at which two or more principal directions coincide. If $s = 0$, let $J_{\rho+1,0}$ be the locus of points at which the tangent $S(\rho)$, which in general is the S_r, reduces to a hyperplane. The construction of $J_{\rho+1,s}$ may break down if Φ satisfies a certain differential equation; it is assumed in what follows that this is not the case.

Let $|C|$ be a simple linear system of curves on an irreducible algebraic surface F, and Φ the transform of F by $|C|$. Corresponding to $J_{\rho+1,s}$ on Φ there is a curve, also denoted by $J_{\rho+1,s}$ on F, which is a birational invariant of $|C|$. The author shows that

$$(*) \quad J_{\rho+1,s} \equiv [s(\rho-s+2)-1]\left\{2\left[\binom{\rho+2}{2}+s\right]C + \left[2\binom{\rho+2}{3}+s(\rho+1)\right]K\right\}, \quad s>0;$$

$$J_{\rho+1,0} \equiv \binom{\rho+2}{2}C + \binom{\rho+2}{3}K;$$

K being a canonical curve of F. The results also apply if $|C|$ is replaced by a linear system of simple integrals of the first kind; one merely removes the terms involving C from (*). Several special cases, for small values of r, are discussed in detail. It is pointed out, without detailed proofs, how certain of the results for linear systems of curves can be extended to systems of V_{k-1}'s on a V_k.

If $|C|$ is taken to be a pluricanonical system, the corresponding $J_{\rho+1,s}$ in S_r is a birational invariant of F and hence

is invariant under any birational transformation (b.t.) (necessarily a homography in S_r) of F into itself. If $J_{\rho+1,s}$ spans S_r, the number of b.t. of F into itself cannot exceed the number of b.t. of $J_{\rho+1,s}$ into itself. The known upper bounds for the latter thus give bounds for the former. This argument can also be applied to linear systems of integrals.

R. J. Walker (Los Angeles, Calif.).

Burau, Werner. Eine gemeinsame Verallgemeinerung aller Veroneseschen und Grassmannschen Mannigfaltigkeiten und die irreduziblen Darstellungen der projektiven Gruppen. Rend. Circ. Mat. Palermo (2) **3**, 244–269 (1954).

If $n, m_1, \cdots, m_\rho$ are any $\rho+1$ integers satisfying

$$n>m_1\geqq\cdots\geqq m_\rho>0,$$

then the aggregate of ordered sets of spaces $X^{(1)}_{m_1}, \cdots, X^{(\rho)}_{m_\rho}$ in S_n (suffixes indicating dimensions) is mapped on the Segre product $G^n_{m_1}\times\cdots\times G^n_{m_\rho}$ of the appropriate Grassmannian manifolds. On this the locus representing sets satisfying the incidence conditions $X^{(1)}_{m_1}\supset\cdots\supset X^{(\rho)}_{m_\rho}$ is denoted by $J_{n;m_1,\cdots,m_\rho}$. The conditions expressing the incidence relations being linear on the coordinates of a point of the Segre product, $J_{n;m_1,\cdots,m_\rho}$ is a linear section of this.

If of the integers $m_1, \cdots, m_\rho$ the first ρ_1 are all equal to n_1, the next ρ_2 to $n_2, \cdots$, and the last ρ_s to n_s, where $n_1>n_2>\cdots>n_s$, $\rho_1+\cdots+\rho_s=\rho$, the incidence conditions require the ρ_i spaces X_{n_i} to coincide, and the Segre product of ρ_i $G^n_{n_i}$'s appears only as its diagonal manifold $V^{\rho_i}(G^n_{n_i})$, i.e. the projective model of the complete system of ρ_i-ic sections of $G^n_{n_i}$. We thus obtain as an image of sets of distinct incident spaces $X^{(1)}_{n_1}\supset\cdots\supset X^{(s)}_{n_s}$ the same manifold as before, but now regarded as existing on the Segre product $V^{\rho_1}(G^n_{n_1})\times\cdots\times V^{\rho_s}(G^n_{n_s})$ and denoted accordingly by $J^{\rho_1,\cdots,\rho_s}_{n;n_1,\cdots,n_s}$. It is clear that $J^{\rho_1,\cdots,\rho_s}_{n;n_1,\cdots,n_s}=J^{\rho_s,\cdots,\rho_1}_{n;\nu_s,\cdots,\nu_1}$, where $\nu_i=n-n_i-1$; moreover, there is in the ambient of $J^{\rho_1,\cdots,\rho_s}_{n;n_1,\cdots,n_s}$ a well defined envelope dual to it, each hyperplane of which represents an incident set of spaces $Y^{(s)}_{\nu_s}\supset\cdots\supset Y^{(1)}_{\nu_1}$ a point of $J^{\rho_1,\cdots,\rho_s}_{n;n_1,\cdots,n_s}$ lying in a hyperplane of the envelope if and only if $X^{(i)}_{n_i}$, $Y^{(i)}_{\nu_i}$ meet in a point $(i=1, \cdots, s)$.

Every collineation in S_n is mapped on a collineation in the ambient of $J^{\rho_1,\cdots,\rho_s}_{n;n_1,\cdots,n_s}$ in which this figure and its dual are invariant; every correlation in S_n by a correlation in which the two figures are interchanged.

The representation thus obtained of the projective group G^n in S_n is irreducible; and every irreducible representation of G^n is identical with this, for some values of $n_1, \cdots, n_s$, $\rho_1, \cdots, \rho_s$. The bulk of the paper is devoted to identifying this last result with Schur's algebraic specification of the irreducible representations of G^n [S.-B. Preuss. Akad. Wiss. 1927, 58–75].

P. Du Val (London).

Burau, Werner. Grundmannigfaltigkeiten der projektiven Geometrie. V, VI. Collect. Math. **6**, 125–220 (1953).

Parts I–IV of this work appeared in Collect. Math. **3**, 53–163 (1950); **5**, 4–118 (1952); MR **13**, 977; **15**, 895. The present two parts are the last. Part V deals with N-manifolds, $N_{\binom{k+2}{k}}$ being defined as the Grassmannian image of the aggregate of self-polar S_k's in the general null correlation in S_{2k+1}. It is a linear section, by a suitable space of dimensions $\binom{2k+2}{k+1}-\binom{2k+2}{k-1}-1$, of the Grassmanian G_k^{2k+1} of all S_k's in S_{2k+1} (i.e. the self polar S_k's are the complete intersection of $\binom{2k+2}{k-1}$ linearly independent linear complexes) but for $k>2$ is of higher dimensions than the general such section, and of course of different order, though no attention seems to be given to this question. $N_{\binom{k+2}{k}}$ contains lines, ∞^k through each point, but no higher linear space; it is its own minimum model.

Part VI deals more or less in parallel with the five types of manifold treated in the previous five sections, developing some properties common to all of them. The group $\mathfrak{G}$ of projective self transformations of each, for instance, is irreducible, not only in the ambient of the manifold itself, but in that of its V^k transform (i.e. the projective model of its complete system of k-ic sections.) Each also is self dual in that it uniquely determines an aggregate of hyperplanes which is dual to the manifold itself as aggregate of points, and the corresponding duality relations in the ambient of the V^k transform of the ambient of the manifold receive some attention. In this last space the k-ic hypersurfaces through the manifold, and in particular the type of imprimitivity of the group induced by $\mathfrak{G}$, is studied in a good many special cases, mostly for $k=2$. This leads to some results on stereographic projection of the various manifolds.

P. Du Val (London).

Galbură, Gh. Sur les variétés canoniques et cycliques, caractéristiques d'une variété algébrique. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 61–64 (1954). (Romanian. Russian and French summaries)

By using a formula of Gamkrelidze [Dokl. Akad. Nauk SSSR (N.S.) **90**, 719–722 (1953); MR **15**, 459] the author states the theorem that for a non-singular algebraic variety the characteristic cycles defined by the reviewer are homologous to the canonical cycles defined by Eger and Todd. The proof is only sketched. For the case that the algebraic variety is a complete intersection of hypersurfaces the same result was proved by Hodge [Proc. London Math. Soc. (3) **1**, 138–151 (1951); MR **13**, 380].

S. Chern.

Boughon, Pierre. Spécialisations et dérivations. Application à la définition du cycle caractéristique d'un diviseur. C. R. Acad. Sci. Paris **240**, 1185–1187 (1955).

L'auteur montre d'abord comment la définition classique "lim $[f(x+h)-f(x)]/h$" des dérivées se transpose au cas abstrait. Soit $k(x)$ une extension régulière de dimension 1 d'un corps k, s un isomorphisme de $k(x)$ tel que $k(x^s)$ soit linéairement disjoint de $k(x)$; on prolonge la spécialisation $(x, x^s)\to(x, x)$ en une place φ de $k(x, x^s)$; si y et t sont des éléments de $k(x)$ tels que y soit entier sur $k[t]$, alors l'application $y\to\varphi(y^s-y/t^s-t)$ est la k-dérivation D de la clôture intégrale de $k(t)$ telle que $Dt=1$. Soit alors $F(u, X)=0$ l'équation de l'élément générique d'une famille (F) de diviseurs de A_n paramétrée par une courbe non singulière de point générique (u) sur k. Pour tout X_v de (F) on prend un diviseur $X_{v'}$ générique sur $k(v)$, et on définit le cycle caractéristique de X_v comme étant la spécialisation de $X_v\cdot X_{v'}$ prolongeant $v'\to v$; celle ci est unique puisque la variété des paramètres est une courbe non singulière. En se servant du résultat ci dessus sur les dérivations on montre que, s'il est défini, le cycle intersection des diviseurs $F(u, X)=0$ et $DF(u, X)=0$ (D: k-dérivation non triviale de $k(u)$) est le cycle caractéristique de X_u.

P. Samuel.

Differential Geometry

Scherk, Peter. Dually differentiable points on plane arcs. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 43–48 (1954).

Let $P(t)$, $p(t)$, $0 \leqq t \leqq 1$, be the points and tangents, respectively, of an arc A in the real projective plane. $P(t)$, $p(t)$ depend continuously on t. The arc A is assumed to be both differentiable and dually differentiable, i.e. secant $P(t)P(t_0) \to p(t_0)$ and $p(t) \cap p(t_0) \to P(t_0)$ as $t \to t_0$. For $\epsilon > 0$, $0 \leqq t_0 \leqq 1$, N_r (N_l) is the set of $P(t)$ for which $t_0 < t < t_0 + \epsilon$, ($t_0 - \epsilon < t < t_0$). Each sufficiently small N_r can be referred to an affine coordinate system (x, y) for which (1) $P(t_0)$ is the origin, (2) $p(t_0)$ is the line $y = 0$, (3) $x > 0$ for all points of N_r, and (4) the slope of each tangent of N_r is positive. For each $P(t_0)$ numbers e_r, e_r^* are defined which have the following properties. $e_r = 1$ ($= 2$) if N_r is in the first (fourth) quadrant for sufficiently small ϵ and otherwise $e_r = \infty$. e_r^* (the dual concept of e_r) $= 1$ ($= 2$) if the tangents of N_r all intersect the positive (negative) x-axis and otherwise $e_r^* = \infty$. e_l, e_l^* are defined in a similar fashion by means of arcs N_l. It is shown for all N_r that (e_r, e_r^*) must be one of the following: (1, 1), (1, 2), (2, 1), (1, ∞), (∞, 1), (∞, ∞). Moreover, for any given one of the above pairs, the author constructs an N_r for which the associated (e_r, e_r^*) is this given value pair. This is the principal result of the paper.

If m is a line for which $P(t_0) \in m$, $p(t_0) \neq m$, then, for sufficiently small N_r, N_l, these arcs are either separated by m or lie on the same side of m. In the first case a_0 is defined to be 1, in the second to be 2. a_0^* is the corresponding number for the dual of A. Thus, for each $P(t_0)$ a characteristic matrix $\begin{pmatrix} e_l & a_0 & e_r \\ e_l^* & a_0^* & e_r^* \end{pmatrix}$ is defined. An arc exists for each of the 144 possible characteristic matrices. Let (a_0, a_1) be the characteristic of A at $P(t_0)$ [cf. Scherk, Časopis Pěst. Mat. Fys. **66**, 172–191 (1937)] and (a_0^*, a_1^*) that of the dual of A at $p(t_0)$. The relationship between the above matrix and $\begin{pmatrix} a_0 & a_1 \\ a_0^* & a_1^* \end{pmatrix}$ is discussed. *D. Derry.*

Scherk, Peter. Elementary points on plane arcs. Trans. Roy. Soc. Canada. Sect. III. (3) **48**, 49–53 (1954).

Let $P(t)$, $0 \leqq t \leqq 1$, be the points of a continuous arc A in the real projective plane. A tangent $p(t_0)$, $0 \leqq t_0 \leqq 1$, is assumed to exist for which secant $P(t_0)P(t) \to p(t_0)$ as $t \to t_0$. If secant $P(t)P(t') \to p(t_0)$ as $t \to t_0$, $t' \to t_0$, then A is said to be strongly differentiable at t_0. If A is strongly differentiable at t_0 and $p(t) \cap p(t') \to P(t_0)$ as $t \to t_0$, $t' \to t_0$, then A is defined to be maximally differentiable at t_0. A number of criteria are given for both types of differentiability some of which involve the characteristic matrices of the paper of the preceding review. Among these are the following. Let $(a_0(t), a_1(t))$ be the characteristic of A at the point $P(t)$. A necessary and sufficient condition that A be strongly differentiable at t_0 is that a curve neighborhood of $P(t_0)$ exists within which $p(t)$ is continuous and $a_0(t) = 1$. A is maximally differentiable at all points if and only if A has local order 2 at each point.

A point $P(t)$ of A is defined to be elementary if right and left curve neighborhoods of $P(t)$ exist both of which are arcs of order 2. A number of necessary and sufficient conditions are given for a point $P(t)$ to be elementary

D. Derry (Vancouver, B. C.).

Arf, C. Sur le théorème de Reiss. Rend. Mat. e Appl. (5) **14**, 181–191 (1954).

Soit C_n une courbe algébrique plane de classe n, P un point quelconque, φ_i ($i = 1, 2, \cdots, n$) l'angle d'une tangente de P à C_n avec une direction fixe, M_i le point de contact, $t_i = PM_i$, r_i le rayon de courbure à M_i. L'expression $\gamma = \sum r_i t_i^{-1} d\varphi_i$ est, d'après le théorème de Reiss, une différentielle totale. L'auteur, suivant une suggestion de Blaschke, calcule $\int_{P_0}^{P} \gamma$ et des intégrales analogues. *O. Bottema.*

Blaschke, Wilhelm. Zur Kinematik und hyperbolischen Geometrie. Rend. Mat. e Appl. (5) **14**, 16–22 (1954).

Verf. betrachtet die Kinematik der Gruppe G_6 der Kreisverwandtschaften in der Ebene. In $x = ax' + b/cx' + d$ sind die Komplexen a, b, c, d von der Zeit t abhängig. Man hat $dx/dt = P + Qx + Rx^2$, die Bewegungsvorgänge hängen somit mit den Riccatischen Differentialgleichungen zusammen. Es gibt in jedem Augenblick zwei Pole der Bewegung. Anwendung von tetrazyklischen Koordinaten gibt den Zusammenhang mit der hyperbolischen Raumgeometrie und mit der ins Komplexe erweiterten Kinematik der elementaren Kugeldrehungen. *O. Bottema* (Delft).

Forte, Bruno. Di alcune proprietà cinematiche riguardanti i moti rigidi di rotolamento. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 468–473 (1954).

Some differential geometrical properties of the fixed and the moving ruled surfaces S and S' of instantaneous screw axes related to the motion of a rigid body in space. The author defines two points, P on S, P' on S', conjugate of order m and proves that a necessary and sufficient condition is that the sphere through P with center P^1 have m-order contact with the trajectory of P. Characterization of pairs of points conjugate of the third and fourth order. Generalization of Savary's theorem. *O. Bottema* (Delft).

Noli, Walter. Über Schraubenabbildungen. Mitt. Math. Sem. Giessen no. **51**, i+66 pp. (1954).

Verf. betrachtet ein Problem aus der Praxis, das sich auf Schraubenmutter und Schraubenspindel bei verschiedenen, jedoch parallelen Achsen bezieht; die Schrauben haben gleiche Parameter und die Abstände der Punkte der Berührungskurve von den zwei Achsen soll konstant sein. Bei der Lösung mittels darstellender Geometrie werden geometrische Sätze gefunden, die dann weiter diskutiert und generalisiert werden. Sind eine Schraubenachse a und der Parameter c gegeben, so ist jedem Punkt P des Raumes eine Tangente t zugeordnet; wie wird t sich ändern wenn a eine Bewegung ausführt? Die Abbildung von a auf t wird "Schraubenabbildung" genannt. Es werden verschiedene Achsenmannigfaltigkeiten und die zugeordneten Tangenten betrachtet. *O. Bottema* (Delft).

Horninger, H. Über Konchoidenschraublinien und die durch Konchoidenschraubung erzeugbaren Regelschraubflächen. Monatsh. Math. **58**, 225–240 (1954).

Fortsetzung einer früheren Arbeit [Monatsh. Math. **58**, 193–212 (1954); MR **16**, 619]. Unter den Trochoidenschraubungen nimmt die Konchoidenschraubung eine Sonderstellung ein, weil durch sie gerade Regelschraubflächen erzeugbar sind. *O. Bottema* (Delft).

Kadeřávek, František. Sur les surfaces avec les courbes de la plus grande inclinaison vers le plan donné et qu'on peut facilement déterminer. Časopis Pěst. Mat. **78**, 341–346 (1953). (Czech)

Dans le travail l'auteur considère la surface d'un cone quadratique déterminé par le sommet dans le plan π et par la parabole dans le plan horizontal $\pi' \| \pi$. Les coupes horizontales de la surface et les courbes de la plus grande in-

clinaison sont les paraboles qui ont les projections orthogonales dans le système des paraboles confocales et coaxiales. On considère aussi les surfaces, p.ex. celle donnée par l'équation

$$-\frac{x^2}{z^2+l^2}+\frac{y^2}{z^2}=1,$$

qui possèdent dans les plans $z=$const. les coniques. Leur projections dans le plan $z=0$ sont les coniques confocales. Les propriétés de la projection des surfaces sont aussi mentionées. *F. Vyčichlo* (Prague).

Vidav, Ivan. Elementare Herleitung einer Flächenformel für Figuren auf der Kugel. Slovenska Akad. Znan. Umet. Razred Mat. Fiz. Tehn. Vede. Ser. A. Razprave **4**, no. 2, 17–28 (1952). (Slovenian. German summary)

A simple and elementary proof of the formula

$$\int_C [\mathbf{r}, \mathbf{n}, d\mathbf{n}]=2n\pi-p,$$

where $\mathbf{r}=\mathbf{r}(t)$ is a closed curve C on a unit sphere S, $\mathbf{n}$ is a vector field orthogonal to $\mathbf{r}$, p the area defined by C on S, and n is an integer. The importance of this formula is shown by different applications. *V. Hlavatý*.

Backes, F. Sur une configuration particulière des douze surfaces de Darboux. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 370–372 (1955).

Nožička, Frant. La courbe et son arc dans l'espace affin. Časopis Pěst. Mat. **78**, 307–339 (1953). (Czech)

Mémoire contient les résultats bien connus de la géométrie différentielle des courbes dans l'espace affin qui sont ici ordonnés dans l'entier instructif. Les théorèmes sont soigneusement prouvés et démontrés aussi par exemples bien choisis. *F. Vyčichlo* (Prague).

Manara, Carlo Felice. Cubica equianarmonica legata ad una terna di E_1. Boll. Un. Mat. Ital. (3) **9**, 353–359 (1954).

Let, in a projective plane S_2, E_1, E_1', E_1'' be three curvilinear elements of the first order, whose tangents at three points A, B, C are a, b, c respectively, and suppose that through each vertex of the triangle ABC passes only one side of the trilateral abc. If A', B', C' are the points of intersection of a, b, c with the lines BC, CA and AB respectively, then the product k of the three ratios of lengths AB/BC', BC/CA', CA/AB' is a projective invariant. For E_1, E_1', E_1'', a rational equiharmonic cubic Φ is determined. By means of k and the Mehmke-Segre invariants J_a, J_b, J_c of Φ and the ∞^3 system of cubics containing E_1, E_1', E_1'' respectively at A, B, C, a triple of curvilinear elements of the second order in S_2 and, particularly, those belonging to a pencil of cubics are characterized. *C. C. Hsiung.*

Muracchini, Luigi. Sulle trasformazioni puntuali fra due S_r che mutano ∞^{r-1} iperpiani in iperpiani. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **9**, 31–37 (1952).

Let $x_i(y_i)$, $i=0, 1, \cdots, r$, be a set of homogeneous projective coordinates of a point in a projective space $S_r'(S_r'')$ of dimension r, and let a point transformation T between the two spaces S_r', S_r'' be analytically represented by

$$x_i=x_i(u_1, \cdots, u_r), \quad y_i=y_i(u_1, \cdots, u_r) \quad (i=0, 1, \cdots, r),$$

where the functions and their derivatives are continuous for all values of u_i in a region R, and the determinants $(x, \partial x/\partial u_1, \cdots, \partial x/\partial u_r)$, $(y, \partial y/\partial u_1, \cdots, \partial y/\partial u_r)$ are not zero in R. It is proved that if the transformation T for $r\geqq 3$ transforms a $(r-1)$-parameter system Σ_{r-1} of hyperplanes in S_r' into a system Σ of hyperplanes in S_r'', and if the system Σ_{r-1} or Σ is not a star in S_r' or S_r'' respectively, then T is a homograph. For the case $r=2$, there are determined some non-homographical transformations between planes which transform a pencil and two one-parameter systems of lines respectively into a pencil and two one-parameter systems of lines. *C. C. Hsiung* (Bethlehem, Pa.).

Blaschke, Wilhelm. Sulla geometria proiettivo differenziale delle superficie V_2 nello spazio P_4. Rend. Circ. Mat. Palermo (2) **3**, 193–197 (1954).

Let $z(u, v)$ be the vector of a general point z on an analytic surface V_2 in a projective space P_4 of four dimensions with respect to a projective homogeneous coordinate system, and suppose that z and its partial derivatives z_u, z_v, z_{uu}, z_{vv} are linearly independent. By a geometric determination of the parameters u and v, the author obtains at the point z some covariant configurations and invariants of the congruence Z generated by the lines of intersection of the tangent planes at points of V_2 with a fixed hyperplane in P_4. Some equivalent conditions for Z to be a W congruence are also derived. *C. C. Hsiung* (Bethlehem, Pa.).

Terracini, Alessandro. Sugli elementi curvilinei composti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88**, 7–15 (1954).

Denote by $E(p_1, \cdots, p_n; m)$ the totality of all curvilinear elements in a projective space S_{n+1} of dimension $n+1$ $(n\geqq 2)$, each of which can be represented in non-homogeneous projective coordinates $x_0=x, x_1, \cdots, x_n$ by a system of power series of the form

$$x_i=\sum_{k=0}^{m} a_{i,i+k}x^{p_i+k}+(p_i+m+1) \quad (a_{ii}\neq 0, 1\leqq i\leqq n),$$

where (p_i+m+1) denotes the terms of degree $\geqq p_i+m+1$ in x, $a_{i,i+k}$ are constants, m is an integer $\geqq 0$, and p_i are integers satisfying $1<p_1<p_2<\cdots<p_n$. In particular, $E(2, 3, \cdots, n+1; m)$ is briefly written as $E_{2,m}$. In two previous papers [Univ. e Politec. Torino. Rend. Sem. Mat. **12**, 265–281 (1953); Boll. Un. Mat. Ital. (3) **8**, 368–374 (1953); MR **15**, 826] the author has obtained some projective invariants, denoted by γ_{ij} and γ^*_{ij} $(1\leqq i\leqq n, 1\leqq j\leqq n)$, for two $E(p_1, p_2, \cdots, p_n; m+1)$ containing a same

$$E(p_1, p_2, \cdots, p_n; m).$$

Two $E_{2,1}$ in S_{n+1} containing a same $E_{2,0}$ are called self-dual if $\gamma_{ij}=\gamma^*_{ij}$ for all i, j $(1\leqq i\leqq n, 1\leqq j\leqq n)$. In this paper pairs of self-dual $E_{2,1}$ are studied by means of polarities.

C. C. Hsiung (Bethlehem, Pa.).

Su, Buchin. The step point of a curve in projective hyperspace. Acta Sci. Sinica **3**, 107–151 (1954).

Let an analytic curve C in a projective space S_n of dimension $n\geqq 2$ be represented by analytic functions $x_i=x_i(t)$ $(i=1, 2, \cdots, n)$. For a given set of positive integers $\alpha_1, \cdots, \alpha_n$, denote

$$\Delta(\alpha_1, \cdots, \alpha_n)=\begin{vmatrix} x_1^{(\alpha_1)} & \cdots & x_n^{(\alpha_1)} \\ x_1^{(\alpha_1+\alpha_2)} & \cdots & x_n^{(\alpha_1+\alpha_2)} \\ \cdot & \cdots & \cdot \\ x_1^{(\alpha_1+\cdots+\alpha_n)} & \cdots & x_n^{(\alpha_1+\cdots+\alpha_n)} \end{vmatrix},$$

where $x_i^{(n)}$ denotes the nth derivative of x_i with respect to t. A point P_0 on C for which $t=t_0$ is said to be of the type

$(\alpha_1, \cdots, \alpha_n)$, if as $t=t_0$, $\Delta(\alpha_1, \cdots, \alpha_n)\neq 0$, but

$$\Delta(\alpha_1', \cdots, \alpha_n')=0$$

for every set of positive integers $\alpha_1', \cdots, \alpha_n'$ provided that either $\alpha_1'<\alpha_1$, or $\alpha_1'=\alpha_1$, $\alpha_2'<\alpha_2$, or $\alpha_1'=\alpha_1$, $\alpha_2'=\alpha_2$, $\alpha_3'<\alpha_3$, $\cdots$, or $\alpha_1'=\alpha_1, \cdots, \alpha'_{n-1}=\alpha_{n-1}$, $\alpha_n'<\alpha_n$. In particular, a point of the type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called a step point of order $s(\geqq 2)$ if $\alpha_1=1, \alpha_2=1, \cdots, \alpha_{s-1}=1, \alpha_s=2, \alpha_{s+1}=1, \cdots, \alpha_n=1$. The purpose of this paper is to study the projective differential geometry of a curve at a step point of order $s(\geqq 2)$ in S_n. A pyramid of reference is geometrically determined, and canonical power series expansions of the curve in the neighborhood of a step point are obtained.

C. C. Hsiung (Bethlehem, Pa.).

Jonas, Hans. Reflexion von Strahlensystemen: Dupin-Darbouxscher Satz, Transformation isometrischer Flächenpaare. Math. Nachr. **12**, 367–383 (1954).

Verf. betrachtet die Reflexion einer Strahlenkongruenz (X_1) an einer Fläche F derart dass die Developpablen (α, β) der zweiten Kongruenz (X_2) mit den Developpablen von X_1 korrespondieren. Ist X_1 und deshalb auch X_2 ein Normalensystem, so entspricht den Developpablen auf F ein konjugiertes Netz. Es gibt aber im allgemeinen nicht von Normalen gebildete Kongruenzen wobei F beliebig ist aber die beiden Kongruenzstrahlen ganz speziell mit F verbunden sind (optische Achsen). Im zweiten Teil der Arbeit wird eine anschauliche Eigenschaft der auf ein isometrisches Flächenpaar ausgeübten biharmonischen Transformationen angegeben.

J. Haantjes (Leiden).

Saban, Giacomo. Su alcune famiglie di geodetiche. Rend. Mat. e Appl. (5) **14**, 115–124 (1954).

Pour une surface réglée (R) d'une congruence rectiligne dont les génératrices s'appuient sur un contour fermé C, l'intégrale $\oint_C \mathbf{g}_1 \cdot d\mathbf{r}$, où $\mathbf{g}_1$ est le verseur de la génératrice conjuguée de la surface et $d\mathbf{r}$ le vecteur représentant un déplacement infinitésimal le long de C, a la même valeur pour tous les contours fermés de (R) entourant la surface réglée (invariant de E. Cartan). L'auteur envisage une famille fermée de ∞^1 géodésiques $C(\lambda)$ d'une surface Σ, et met en évidence, pour une telle famille, l'invariant intégral

$$I=\oint_\gamma \frac{\partial \mathbf{r}(s,\lambda)}{\partial s}\cdot\frac{\partial \mathbf{r}(s,\lambda)}{\partial \lambda}d\lambda,$$

où γ est une courbe fermée quelconque entourant la famille $C(\lambda)$, s l'arc de l'une quelconque des courbes de la famille, et λ le paramètre fixant la courbe dans la famille considérée. Il montre que, comme pour l'invariant intégral de E. Cartan, la valeur de I (qu'il appelle le pas de Σ) est la longueur de l'arc déterminé sur une même géodésique $C(\lambda)$ par une trajectoire orthogonale des $C(\lambda)$ ayant son point de départ et d'arrivée sur la géodésique envisagée, et il donne diverses applications de l'existence de l'invariant I.

P. Vincensini (Marseille).

Saban, Giacomo. Sur les congruences cylindriques. Rev. Fac. Sci. Istanbul. Sér. A. **19**, 108–118 (1954). (Turkish summary)

L'auteur développe une communication faite au congrès international des mathématiciens d'Amsterdam (2–9 Sept. 1954). Il rappelle les principes fondamentaux de la représentation duale des congruences rectilignes, et les applique à l'étude des congruences cylindriques, c'est-à-dire dont la représentation sphérique, au lieu de remplir une aire de la sphère image, se réduit à une courbe de cette sphère. Il établit ainsi une suite de résultats, en soulignant le rôle spécial que jouent, dans la théorie, les congruences (qu'il désigne par congruences-p) dont les cylindres se réduisent à des plans, ainsi que certaines congruences-p particulières, dites congruences-d, dont il donne diverses propriétés intéressantes, faisant notamment intervenir les points centraux et les paramètres de distribution des surfaces réglées issues des différents rayons. La condition de coïncidence, sur chaque rayon d'une congruence cylindrique, des points centraux des différentes surfaces réglées le contenant, conduit aux congruences synectiques, qui sont d'ailleurs les seules congruences cylindriques normales.

P. Vincensini (Marseille).

Godeaux, Lucien. Alcune osservazioni sulle congruenze W. Univ. e Politec. Torino. Rend. Sem. Mat. **13**, 39–46 (1954).

(x) étant une surface rapportée à ses asymptotiques (u, v), l'auteur envisage la suite de Laplace L à laquelle appartiennent les images U, V des tangentes asymptotiques de (x) sur l'hyperquadrique de Klein de S_5. Si L se termine dans le sens v au point U_n en présentant le cas de Laplace, elle se termine aussi dans le sens u en présentant le cas de Goursat. $n=0$ correspond au cas où (x) est réglée; pour $n=1$ les asymptotiques (u) de (x) appartiennent à des complexes linéaires, et pour $n=2$ les réglées gauches asymptotiques le long des courbes u appartiennent à des complexes linéaires. En considérant une congruence W dont (x) est une nappe focale, et en désignant par $(\bar{x})$ l'autre nappe focale, l'étude de la suite $\bar{L}$, analogue à L et relative à $(\bar{x})$, montre que dans le cas $n=0$ les asymptotiques u de $(\bar{x})$ appartiennent à des complexes linéaires, et que dans le cas $n=1$ les réglées gauches asymptotiques le long des courbes u de $(\bar{x})$ appartiennent à des complexes linéaires. Si F_0 désigne une surface réglée dont les génératrices rectilignes sont les courbes u, F_1 une surface dont les asymptotiques u appartiennent à des complexes linéaires, et F_2 une surface dont les réglées gauches asymptotiques le long des courbes u appartiennent à des complexes linéaires, l'auteur établit ainsi que si une congruence W a pour nappe focale une F_1, l'autre nappe focale est en général une F_2 (exceptionnellement une F_0 ou une F_1), et il signale l'intérêt que présenterait une étude des surfaces F_2.

P. Vincensini.

Godeaux, Lucien. Sur la théorie des congruences W. Acad. Roy. Belg. Bull. Cl. Sci. (5) **40**, 1028–1037 (1954).

Dans des publications antérieures [La théorie des surfaces et l'espace réglé, Hermann, Paris, 1934; Acad. Roy. Belg. Bull. Cl. Sci. (5), **40**, 880–885 (1954); MR **16**, 513], l'auteur a montré l'intérêt qu'il y a à attacher à une congruence W (de rayon générateur j) les quatre suites de Laplace suivantes: les deux suites contenant, respectivement, les images sur l'hyperquadrique de Klein de S_5 des tangentes asymptotiques des nappes focales de (j); la suite déterminée par l'image J de la génératrice j de (j), et la suite issue du point P seconde image du complexe linéaire osculateur à (j) suivant j. En vue de nouvelles recherches, il établit les formules relatives à ces quatre suites de Laplace, et il rattache, à ces formules, une condition nécessaire et suffisant pour qu'une congruence W appartienne à un complexe linéaire.

P. Vincensini (Marseille).

Godeaux, Lucien. Sur la théorie des congruences W. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 343–345 (1955).

Comme complément à notre première note [analysée ci-dessus], nous montrons comment on peut obtenir la seconde surface focale d'une congruence W lorsque l'on connaît la première.

Résumé de l'auteur.

Shri Krishna. Correspondence of certain curves on the focal surfaces of a rectilinear congruence. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19**, 69–78 (1954). (Turkish summary)

L'auteur étudie les circonstances géométriques qui accompagnent certaines correspondances entre lignes remarquables tracées sur les deux nappes de la surface focale d'une congruence rectiligne. Si les lignes de longueur nulle se correspondent, ou bien la congruence ne peut pas être représentée sur une sphère réelle, ou bien les lignes paramétriques (orthogonales) de la représentation sphérique ont numériquement la même courbure géodésique. La correspondance des lignes caractéristiques sur les deux nappes focales exige la correspondance des lignes asymptotiques et des lignes de courbure, et la correspondance des lignes de courbure entraîne celle des asymptotiques et des caractéristiques. Le cas des congruences pseudosphériques est envisagé plus spécialement, et donne lieu à l'énoncé d'un certain nombre de propriétés.

P. Vincensini.

Gheorghiev, Gh. Quelques problèmes géométriques relatifs à un champ de vecteurs unitaires. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 101–123 (1954). (Romanian. Russian and French summaries)

L'auteur étudie des questions relatifs à un champ de vecteurs unitaires et à la variété orthogonale anholonome associée V_{2n}. L'instrument de recherches est la méthode du repère mobile telle qu'elle est exposée par S. P. Finikoff [Metod vnešnih form Kartana . . . , OGIZ, Moscow-Leningrad, 1948, Cap. XIV; Teoriya kongruènciĭ, Gostehizdat, Moscow-Leningrad, 1950, pp. 68, 518; MR **11**, 597; **12**, 744]. Après l'introduction des notions fondamentaux, il détermine les trièdres de Frenet des courbes de la congruence du champ, puis ceux formés par les tangentes aux courbes du réseau distributif (ou de son bissecteur) de V_{2n} et le vecteur correspondant. Il envisage le comportement de ces trièdres pour les congruences rectilignes et en particulier pour les congruences isotropes, il détermine de deux façons différentes les invariants du champ, et montre que les trois invariants différentiels du champ coïncident avec trois de ses invariants algébriques. Il considère ensuite, sur V_{2n}, quelques systèmes de courbes en involution ou en correspondance biquadratique symétrique. Dans le premier cas on obtient les réseaux réalisant l'égalité de la courbure normale, de la torsion géodésique ou du rapport de ces deux quantités. Dans le deuxième cas les systèmes envisagés sont ceux pour lesquels s'annulent les sommes des courbures normales ou des torsions géodésiques, ou encore le rapport des courbures normales et des torsions géodésiques. Certaines courbes particulières de V_{2n} sont étudiées plus spécialement. Les éléments invariants (Cesàro) des champs des vecteurs d'un trièdre font l'objet d'un examen poussé, conduisant à divers résultats géométriques intéressants.

P. Vincensini (Marseille).

Stoka, Marius I. Sur la géométrie goniométrique des familles de cercles. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 403–412 (1954). (Romanian. Russian and French summaries)

Se basant sur une formule de Kagan relative à l'angle de deux cercles d'un plan, l'auteur établit, par une projection stéréographique de la sphère unitaire sur le plan équatorial, une correspondance biunivoque entre les surfaces de l'espace euclidien ordinaire et les familles de cercles à deux paramètres du plan; puis il montre qu'il n'existe aucune famille de cercles à deux paramètres dont tous les éléments sont tangents entre-eux, et que tout cercle d'une famille plane à deux paramètres appartient à deux familles à un paramètre incluses dans la famille ∞^3 envisagée. Il établit aussi qu'étant donné un cercle (droite) d'une famille à deux paramètres, les cercles de la famille ne le coupant pas forment une famille ∞^2. Partant toujours d'une famille de cercles à deux paramètres, et introduisant la notion d'angle de deux cercles d'une telle famille le long d'une famille à un paramètre, il définit la notion de famille géodésique de la famille à deux paramètres envisagée. *P. Vincensini* (Marseille).

Legrain-Pissard, [N.]. Sur les réseaux homaloïdaux de courbes. Bull. Soc. Roy. Sci. Liège **23**, 426–434 (1954).

Dou, A. Rang der ebenen 4-Gewebe. Abh. Math. Sem. Univ. Hamburg **19**, 149–157 (1955).

In a previous paper [Mem. Real Acad. Ci. Art. Barcelona **31**, 133–218 (1953); MR **16**, 400], the author established criteria for the rank of a plane four-web T^4. They were based on the canonical expansion or symmetric representation of T^4. Identical or similar conditions are discussed in Parts I and III of the present paper. In Part II, T^4 is given in the form $s=s(u,v)$, $t=t(u,v)$. Then the rank of T^4 is positive if and only if a certain differential equation of the form

$$A\frac{dU}{du}+B\frac{d^2U}{du^2}+C\frac{dV}{dv}+D\frac{d^2V}{dv^2}=0$$

has a solution $U=U(u)$, $V=V(v)$. *P. Scherk.*

Chern, Shiing-Shen. La géométrie des sous-variétés d'un espace euclidien à plusieurs dimensions. Enseignement Math. **40** (1951–1954), 26–46 (1955).

This paper is an exposition as of June 1953 of many of the known results in the theory of compact, differentiable manifolds, M^n, imbedded in a Euclidean space, E^{n+N}. The first topic included is the theory of the homomorphisms between the homology and cohomology groups of M and those of the corresponding Grassmann varieties $G(n, N)$ and $G(N, n)$. This material includes the characteristic classes of Stiefel-Whitney and Pontrjagin, and the Gauss-Bonnet theorem. The work of Fary and Milnor on knotted curves is generalized and the total integral curvature of M is defined in a new way. If this curvature is less than twice the area of a unit n-sphere, it is shown that the Betti numbers mod 2 of dimensions $1, 2, \cdots, n-1$ are all zero. Theorems of Tompkins, Kuiper, Chern, and Allendoerfer on rigidity are discussed. Special results for $M^2 \subset E^4$ due to Blaschke and Wen-Tsun Wu, and a list of some unsolved problems complete the paper. *C. B. Allendoerfer* (Seattle, Wash.).

Avez, André. Définition des variétés complètes à métriques indéfinies. C. R. Acad. Sci. Paris **240**, 485–487 (1955).

In order to overcome the difficulty of the existence of isotropic directions in Riemannian spaces with indefinite

metrics the author tries in the case of signature $n-1$ to replace the given metric $g_{\alpha\beta}$ by $-g_{\alpha\beta}+2u_\alpha u_\beta$, where u_α is a time-like unit vector of a certain class $\bar{T}$. Completeness of the given manifold V_n with respect to one such new metric is equivalent to completeness with respect to another one. The class $\bar{T}$ consists of those continuous regular vector fields $u_\alpha(x)$ such that there is no field $\epsilon_\alpha(x)$ and no sequence of points x_i, $i=1, 2, \cdots$, (not in a compact set) with:

$$N[u_\alpha(x)+\epsilon_\alpha(x)]=0, \quad \lim \epsilon_\alpha(x_i)\,|N[u(x_i)]|^{-1/2}=0.$$

The author does not show that the second condition is independent of the coordinate system, nor does he compare his definition of completeness with that of Ehresmann for spaces with affine connections. *A. Nijenhuis.*

Frölicher, Alfred. Zur Differentialgeometrie der komplexen Strukturen. Math. Ann. **129**, 50–95 (1955).

This is the detailed account of work which the author carried out under the supervision of B. Eckmann and of which the main results have been announced in several earlier notes [MR **13**, 75; **15**, 649; **16**, 518]. The paper is divided into five chapters. Ch. I is concerned with general results and conventions on complex and almost complex manifolds. Ch. II defines the torsion tensor. It is proved that its vanishing is a necessary condition for an almost complex structure to be complex and that the condition is sufficient, if the almost complex structure is real analytic. This condition is applied to prove that the almost complex structure on the six-dimensional sphere defined by the Cayley numbers is not complex. These results were found independently by Ehresmann and Libermann [C. R. Acad. Sci. Paris **232**, 1281–1283 (1951); MR **12**, 749].

Ch. III studies the relationship between almost complex structures and affine connections. It is proved that to any almost complex structure there always exist affine connections relative to which the almost complex structure is parallel and that, moreover, there exist symmetric affine connections relative to which the almost complex structure is parallel, if and only if the latter is without torsion. There exists a Riemannian metric relative to which an almost complex structure is parallel, if and only if the latter is without torsion and admits a Kähler metric.

Ch. IV studies the almost complex structure by means of infinitesimal transformations. In particular, the absence of torsion can be expressed in a simple way in terms of the Poisson parenthesis of infinitesimal transformations. This formulation of the torsion tensor is useful in Ch. V, which is devoted to the study of homogeneous almost complex or complex structures. The main result of Ch. V consists in the formulation of a condition for the existence of a homogeneous complex structure on a homogeneous space in terms of the Lie algebra and the subalgebra of the group of isotropy. As a consequence it is proved that a compact connected Lie group of even dimension has a homogeneous complex structure invariant under the left translations. This last result has also been proved by Wang [Amer. J. Math. **76**, 1–32 (1954); MR **16**, 518] and Samelson [Portugal. Math. **12**, 129–132 (1953); MR **15**, 505]. In fact, Wang's paper contains a description of all the compact homogeneous complex manifolds with a finite fundamental group.

S. Chern (Chicago, Ill.).

Lelong-Ferrand, Jacqueline. Groupes d'isometries et formes harmoniques décomposables. C. R. Acad. Sci. Paris **240**, 835–837 (1955).

This note deals with the case in which the infinitesimal group of isometries of the Riemannian manifold V^N is Abelian and its generators $\xi_\alpha^i\partial/\partial x^i$ ($\alpha=1, \cdots, \nu$) are such that there are ν linearly independent harmonic fields of the form $k^{ij}g_{ij}\xi_\alpha^i dx^j$. It is shown that in this case every harmonic form of V^N is the direct sum of pure harmonic forms of different types. This result has applications to the determination of the Betti numbers of varieties V^N with a group of isometries with the given properties, and, in particular, applications are made to varieties whose group of isometries is isomorphic to a torus T^ν. *W. V. D. Hodge.*

Norguet, François. Sur le produit de composition des courants et le nombre algébrique d'intersections de deux chaines. C. R. Acad. Sci. Paris **240**, 830–832 (1955).

The product of composition of currents is generated in the following way. Let U, V, W be three orientable C^∞ manifolds, f a C^∞ application of $U\times V$ to W; if S is a current on U, T a current on V, $S\otimes T$ their tensor product on $U\times V$, their generalised product is $S\bar{*}T=f(S\otimes T)$. Various applications are made. In particular, if for each $x\in U$, $f_x(y)=f(x, y)$ is a C^∞ homeomorphism of V to W, $\mathfrak{T}_{z_0}(x)=f_x^{-1}(z_0)$ is a C^∞ application of U to V for each $z_0\in W$; then if the sum of the dimensions of S and T is equal to dim V=dim W, $S\bar{*}T$ is equal to the algebraic intersection number of $\mathfrak{T}_{z_0}S$ and T at any point z_0 of W (provided this number is defined). *W. V. D. Hodge.*

Blanchard, André. Espaces fibrés analytiques-réels et formes de Kähler analytiques réelles sur les espaces fibrés analytiques complexes. C. R. Acad. Sci. Paris **240**, 1300–1302 (1955).

Proofs of the following theorems are sketched. Theorem 1. Let B be a compact real analytic sub-manifold in R^n, $P(B, G)$ be a real analytic principal fiber bundle whose fiber G is a connected Lie group, and K be a compact subgroup of G. Then the quotient space P/K admits an analytic cross-section whenever it admits a continuous cross-section. Theorem 2. Let $E(B, F)$ be a compact complex Kählerian fiber space whose structural group has only a finite number of connected components. Then a necessary and sufficient condition that E admits a real analytic Kähler form is that B and F admit real analytic Kähler forms. *S. Chern.*

Kodaira, K., and Spencer, D. C. On a theorem of Lefschetz and the lemma of Enriques-Severi-Zariski. Proc. Nat. Acad. Sci. U. S. A. **39**, 1273–1278 (1953).

Soient V une variété kählérienne compacte, de dimension complexe n et S un ensemble analytique principal de V, sans singularité. Soient F un espace fibré analytique complexe à fibre vectorielle de dimension un, F_S la restriction de F à S et $\Omega^p(F)$ (resp. $\Omega^p(F_S)$) le faisceau des germes de p-formes différentielles holomorphes à coefficients dans F sur V (resp. dans F_S sur S). On désigne par $\{S\}$ un espace fibré à fibre vectorielle de dimension un défini par la donnée de S et par $c(F)$ la classe caractéristique de F. On utilise les notations de Kodaira [mêmes Proc. **39**, 1268–1273 (1953); MR **16**, 618]. Théorème 1. Si chaque classe de cohomologie $c(m\{S\}-F)$, $m=1, 2, \cdots, n$, contient une $(1, 1)$-forme d-fermée, réelle $\gamma_m>0$ pour laquelle $\Theta^{p,q}(\gamma_m, u, z)>0$ en chaque point $z\in V$, avec $1\leqq p, q\leqq n$, alors l'homomorphisme $H^q(V, \Omega^p(F))\to H^q(S, \Omega^p(F_S))$ défini par la restriction de F à S est un monomorphisme ou un isomorphisme suivant que $p+q=n-1$ ou $p+q<n-1$. La démonstration utilise le théorème 1 de la Note citée, appliqué à l'espace fibré $F-m\{S\}$. Cela généralise le théorème 2 (dû à Lefschetz quand V est algébrique [L'analysis situs et la géométrie

algébrique, Gauthier-Villars, Paris, 1924, pp. 88–91]). Si $\{S\}$ est suffisamment ample (i.e. si $c(\{S\})$ contient une $(1,1)$-forme d-formée, réelle $\gamma>0$ pour laquelle $\Theta^{p,q}(\gamma,u,z)>0$ en chaque point $z\in V$ $(1\leqq p;q\leqq n)$), alors $H^s(V,C)\to H^s(S,C)$ est un monomorphisme ou un isomorphisme suivant que $s=n-1$ ou $s<n-1$. Etant donné un diviseur quelconque D sur une variété algébrique projective, sans singularité V, en prenant $F=\{D\}$, pour S la section de V par une hypersurface générale d'ordre suffisamment grand et $q=p=0$, le théorème 1 se réduit au lemme d'Enriques-Severi-Zariski [Zariski, Ann. of Math. (2) **55**, 552–592 (1952); MR **14**, 80]. Cette Note contient, de plus, un théorème sur le défaut caractéristique d'un diviseur premier non singulier et une mise au point concernant deux Notes antérieures [Kodaira et Spencer, Proc. Nat. Acad. Sci. U. S. A. **39**, 641–649 (1953); Spencer, ibid. **39**, 660–669 (1953); MR **16**, 617, 618].

P. Dolbeault (Paris).

Yano, Kentaro. Geometria conforme in varietà quasi hermitiane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 449–454 (1954).

A Riemannian space of dimension $2n$, provided with an almost complex structure (i.e. a tensor field F_i^j with $F_i^jF_j^k=-\delta_i^k$) is said to be almost-Hermitian if the metric tensor g_{ij} and the field F_j^k are related by $g_{ij}F_k^iF_l^j=g_{kl}$. The space is called almost-Kählerian if the form

$$\omega=F_k^ig_{ij}dx^j\wedge dx^k$$

is closed, i.e. if $F_{jkl}\equiv 3\partial_{[l}F_{jk]}=0$. The space is called pseudo-Hermitian if the tensor

$$N^i{}_{jk}\equiv 2F_{[k}^a(\partial_{|a|}F_{j]}^i-\partial_{[a|}F_{j]}^t)$$

vanishes. If both $F_{jkl}=0$ and $N^i{}_{jk}=0$, the space is called pseudo-Kählerian. The author studies the invariants of the given fields under the conformal transformations:

$$g_{ij}\to\bar g_{ij}=\rho^2g_{ij},\quad F_i^j\to\bar F_i^j=F_i^j,$$

which implies $F_{ij}\to\bar F_{ij}=\rho^2F_{ij}$. Besides the tensor $N^k{}_{ij}$ and the Weyl tensor $C^i{}_{jkl}$ several others are derived:

$$C^i{}_{ij}=\left(F_{ijk}-\frac{3}{2(n-1)}F_{ij}F_k\right)g^{kl},$$

$$C_{jk}=-2\partial_{[k}F_{j]},$$

$$C_{ijkl}=\gamma_{ijkl}+\gamma_{klij}+\gamma_{ijlk}-\gamma_{jkli},$$

where $F_i=F_{ijk}F^{jk}$, $\gamma_{ijkl}=\partial_lC_{ijk}-\frac{1}{2}(n-1)^{-1}C_{ijk}F_l$. For $2n=4$, $C^i{}_{ij}$ vanishes, and for $2n>4$, $C_{ij}=0$ follows from $C^i{}_{ij}=0$. The following theorem is proved: An almost-Hermitian space is (locally) conformally equivalent to an almost-Kählerian space if and only if for $2n>4$: $C^k{}_{ij}=0$, and for $2n=4$: $C_{ij}=0$. The conditions are also expressed in terms of the torsion of an associated connection, as was previously done by Westlake [Proc. Amer. Math. Soc. **5**, 301–303 (1954); MR **15**, 743]. *A. Nijenhuis* (Princeton, N. J.).

Yano, Kentaro. On three remarkable affine connexions in almost Hermitian spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 24–32 (1955).

The notation and terminology are the same as in the preceding review. Since the fields F_i^j and g_{ij} do not uniquely determine a linear connection for which they are covariant constant, one turns to the study special cases among those that give zero covariant derivatives for F_i^j and g_{ij}. In the present paper three connections are given which, either for the almost-Hermitian spaces, or only for the pseudo-Hermitian case, have the desired property. If $\{{}_i^h{}_j\}$ denotes the Riemannian connection of g_{ij}, then a general connection is defined by $\Gamma^h{}_{ji}=\{{}_i^h{}_j\}+T^h{}_{ji}$. By imposing conditions of symmetry on $T^i{}_{jk}F_{ih}$ the connections become uniquely determined. Of the resulting connections

$$\overset{1}{T}{}_{jih}=\tfrac{1}{4}N_{hij}-\tfrac{1}{2}F_h{}^lF_{ijl},$$

$$\overset{2}{T}{}_{jih}=\tfrac{1}{4}N_{hij}-\tfrac{1}{2}F_j{}^lF_{ihl},$$

$$\overset{3}{T}{}_{jih}=\tfrac{1}{4}N_{jih}+\tfrac{1}{2}F_j{}^mF_i{}^lF_h{}^kF_{mlk},$$

only the first one gives covariant constant g_{ij} without assuming $N_{hij}=0$. This first one was previously established by Lichnerowicz [Arch. Math. **5**, 207–215 (1954); MR **16**, 520], and has the property that it can be obtained from the Riemannian connection by projection of the covariant differential of a (complex) vector v^j of the space E_+ (eigenspace of vectors u^j satisfying $u^jF_j^k=+i\cdot u^k$) on this same space.

The second connection is one which, when restricted to Hermitian spaces, gives the same mapping for the spaces $E_\pm$ as the one that is induced by the canonical projections in the extended space in which the coordinates z and $\zeta=\bar z$ are considered as independent. It was previously considered by Schouten and van Dantzig, by Chern and by Libermann. The third connection has the characteristic that the tensor $S^k{}_{ij}g_{lk}$ is a trivector ($S^k{}_{ij}$ is the torsion of the connection).

In the proof of the theorem that an almost-Hermitian space is pseudo-Kählerian if and only if $\overset{*}{\nabla}_jF_{ik}=0$ ($\overset{*}{\nabla}$=covariant differentiation with respect to the Riemannian connection), due to Yano and Mogi [C. R. Acad. Sci. Paris **237**, 962–964 (1953); MR **15**, 350] the fact is used that $\overset{*}{\nabla}_jF_{ik}$ is determined by N_{ijk} and F_{ijk}, as follows:

$$2\overset{*}{\nabla}_kF_{il}=F_i{}^jN_{jlk}+F_{klj}+F_i{}^mF_l{}^jF_{jkm}.$$

A. Nijenhuis (Princeton, N. J.).

Schouten, J. A., and Yano, K. On an intrinsic connexion in an X_{2n} with an almost Hermitian structure. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 1–9 (1955).

The notation and terminology are the same as those in the two preceding reviews. An exposition of a decomposition process of tensors of the tangent space in parts belonging to the subspaces E_+ and E_- is followed by a restatement of the integration theorem of Eckmann and Frölicher [C. R. Acad. Sci. Paris **232**, 2284–2286 (1951); MR **13**, 75] in terms of the vanishing of $N^k{}_{ij}$, and by a few other conditions equivalent to $N^k{}_{ij}=0$.

The authors proceed to show that there is precisely one affine connection associated with an almost-Hermitian structure which has the following properties: 1) the metric tensor is covariant constant; 2) the field F_i^j is covariant constant; 3) in each 2-direction spanned by vectors v^i and $v^jF_j^i$ (so-called invariant 2-directions) there exist infinitesimal parallelograms. The resulting connection is found to be determined by

$$T_{ijk}=\tfrac{1}{2}(\overset{*}{\nabla}_kF_{ij})F_i{}^l-\tfrac{1}{2}(\overset{*}{\nabla}_lF_{ij})F_k{}^l-\tfrac{1}{2}(\overset{*}{\nabla}_jF_{il})F_k{}^l-\tfrac{1}{4}N_{kij}.$$

The following interpretations are given of the three connections in Yano's paper (reviewed above). If a vector u^i is displaced by the Riemann connection from x to $x+dx$, and if $F_j{}^iu^j$ is displaced by means of the first connection, then the resulting vectors should be incident for all u^i and dx. For the second connection the effect of this connection and the Riemannian one on $F_{ij}u^j$ should be the same for a displacement from x to $x+\epsilon u$, and for the third one the

geodesics of both connections should coincide and have the same parametrizations.

The tensor N derived from F is a special case of a concomitant for any tensor field h_i^j without algebraic restrictions on h:

$$H^h_{ji}=2h_{[j}{}^l\partial_{[l]}h_{i]}{}^h-2h_l{}^h\partial_{[j}h_{i]}{}^l.$$

The form for H, was erroneously given by the authors to be the same as that of N above, but N is a tensor only if $h_i^j h_j^k=\text{const.}$ [cf. Nijenhuis, same Proc. **54**, 200–212 (1951); MR **13**, 281]. However, the tensor N was used before the more general form for H above had been established.

A. Nijenhuis (Princeton, N. J.).

Yano, Kentaro. Sur la correspondance projective entre deux espaces pseudo-hermitiens. C. R. Acad. Sci. Paris **239**, 1346–1348 (1954).

The notation and terminology are the same as in the three papers reviewed above. Given two spaces with pseudo-Hermitian structures, and with connections Γ and Γ' respectively, the author investigates under what circumstances the relation between Γ and Γ' is a projective one:

$$\Gamma'^h_{(ij)}=\Gamma^h_{(ij)}+p_jA_i^h+p_iA_j^h,$$

so that the geodesics of both connections are the same. He restricts himself to the case where the tensor F_i^j is the same for both spaces. Then the fact that F_i^j should be covariant constant in both connections imposes some restrictions on p_i. It appears that there is a projective relation if the tensor

$$2nS^h_{ji}F_i^l-(S^m_{mi}+S^r_{qp}F_r^qF_i^p)A_j^hF_i^l \\ -2nS^l_{ji}F_l^h+(S^m_{mi}+S^r_{qp}F_r^qF_i^p)F_j^h$$

is the same for both spaces.—By checking this relation for any of the three connections given in the paper reviewed second above (or any other: the fourth, of Schouten and Yano, was not yet known when this paper was written), one can verify if a projective relation exists. For Kählerian spaces one has $p_i=0$, and the relation is an affine one.

A. Nijenhuis (Princeton, N. J.).

Yano, Kentaro, and Mogi, Isamu. On real representations of Kaehlerian manifolds. Ann. of Math. (2) **61**, 170–189 (1955).

Ce papier est principalement consacré aux variétés kähleriennes à courbure holomorphique constante. Après avoir rappelé les faits bien connus de la géométrie différentielle locale des variétés complexes et des variétés kähleriennes, les A. étudient les variétés kähleriennes V_{2n} pour lesquelles la courbure k associée à un 2-plan défini par un vecteur et son image par l'opérateur presque complexe (2-plan holomorphique) est indépendante du choix du vecteur; k est alors, en vertu des identités de Bianchi, une constante sur tout l'espace [Bochner, Bull. Amer. Math. Soc. **53**, 179–195 (1947); MR **8**, 490] et une forme explicite du tenseur de courbure est donnée, valable en repères réels arbitraires. La courbure relative à un 2-plan arbitraire est comprise entre $k/4$ et k. Pour qu'il existe sur une variété kählerienne V_{2n} une 2-surface totalement géodésique tangente à tout 2-plan holomorphique, il faut et il suffit que V_{2n} soit à courbure holomorphique constante; de même ces variétés peuvent être caractérisées par la "libre mobilité holomorphique" qui généralise d'une manière évidente la libre mobilité ordinaire. Pour une telle variété à courbure holomorphique constante $k>0$, la distance entre deux points conjugués consécutifs d'une géodésique est égale à $2\pi k^{-1/2}$. Quelques considérations sur les 1-formes harmoniques et les 1-formes de Killing terminent ce travail.

A. Lichnerowicz (Paris).

Yano, K., and Davies, E. T. On the connection in Finsler space as an induced connection. Rend. Circ. Mat. Palermo (2) **3** (1954), 409–417 (1955).

A. Deicke has shown [J. London Math. Soc. **30**, 53–58 (1955); MR **16**, 626] that it is not in general possible to regard an n-dimensional Finsler space (in the sense of Cartan) as a non-holonomic subspace of a Riemannian space of higher dimension without torsion, but that it is always possible to determine a suitable metric and torsion. On the basis of their earlier work on the tensor calculus of contact transformations [Ann. Mat. Pura Appl. (4) **37**, 1–36 (1954); MR **16**, 626] the authors show how the extended point transformations which are the transformations used in Finsler geometry suggest a specialisation of the non-holonomic frames which leads naturally to coefficients of a non-holonomic affine connection in an enveloping space of dimension $2n$. The Finsler connection is then obtained as the corresponding induced connection. The conditions imposed on the torsion tensor appear to be less restrictive than in the work of Deicke.

H. Rund (Toronto, Ont.).

Sasayama, Hiroyoshi. On the extended harmonic and invariant multiple integrals of higher order. Tensor (N.S.) **4**, 122–127 (1954).

The extended multiple integral in the space of higher order was treated by the author for a base space without a Riemann metric [Tensor (N.S.) **2**, 36–46 (1952); **3**, 123–127 (1954); MR **14**, 690; **16**, 402]. In the present paper he extends the harmonic and invariant integrals in the sense of Hodge, de Rham, and Kodaira in a space with Riemann metric and a closed semi-simple finite continuous Lie group of transformations, by means of Katsurada's excovariant differentiation [J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 17–28 (1951); MR **15**, 998] and extended Lie derivative [Tensor (N.S.) **2**, 15–26 (1952); MR **15**, 987]. In addition to some formulas for the extended Laplacian the results are stated that the harmonic property and the invariance of a multiple integral on the base space are equivalent to the ones of the derived integral on the space of line elements of higher order, that both the extended exterior and dual differentials of an extended invariant multiple integral of higher order are invariant and that the extended Lie derivative of an extended harmonic multiple integral of higher order is also harmonic. (From the author's summary.)

A. Nijenhuis (Princeton, N. J.).

Sasayama, Hiroyoshi. On the generalized invariant differential form on manifolds with general connection. Tôhoku Math. J. (2) **5**, 122–127 (1953).

L'A. se propose d'étendre aux variétés générales à connexion, dans le sens de Chevalley, les résultats concernant, pour les variétés de dimension finie, les formes invariantes par le groupe d'holonomie et le premier théorème de Cartan sur les espaces homogènes.

L'A. considère une variété générale X de classe C^k ($k\geq 2$) admettant un espace de Banach C comme espace de coordonnée locale. A chaque point p de X est associée une variété générale Σ_p (dite variété associée) de classe C^k, admettant C comme espace de coordonnée locale et un groupe topologique G comme groupe de transformations. Dans Σ_p un point p^* est choisi qui est dit le centre de Σ_p.

On se donne enfin un ensemble E de transformations Γ: $\Sigma_p \to \Sigma_q$, ou déplacements, satisfaisant aux axiomes des connexions (si Γ et Γ' appliquent Σ_p sur Σ_q, $\Gamma^{-1}\Gamma' \in G$; si Γ_1: $\Sigma_p \to \Sigma_q$ et Γ_2: $\Sigma_q \to \Sigma_r$, $\Gamma_2\Gamma_1 \in E$: $\Sigma_p \to \Sigma_r$; si $\Gamma \in E$, $\Gamma^{-1} \in E$; pour chaque couple p, q de points de X, il existe un Γ appliquant Σ_p sur Σ_q). E est muni d'une structure de pseudo groupe; par chaque $p \in X$, l'ensemble H_p des I de E appliquant $\Sigma_p \to \Sigma_p$, forme un groupe, le groupe général d'holonomie de X en p. L'A. définit de la manière usuelle l'espace vectoriel V_p tangent à une variété X en p. Si $p = p^*$ et si V_p coïncide avec l'espace V_p^* tangent en p^* à Σ_p, Σ_p est dite une variété tangente et p le point de contact correspondant. Nous supposons désormais qu'il en est ainsi et que, si Σ est non-linéaire, chaque point de contact est invariant par déplacement. Le groupe général d'holonomie induit sur V_p un groupe linéaire d'holonomie dH_p. Le cas des espaces homogènes généraux W est un cas particulier du schéma précédent avec $\Sigma_p = W$, $p = p^*$, $E = G$, les groupes d'holonomie devenant les groupes d'isotropie.

Sur X, une r-V-forme est une forme différentielle r-linéaire à valeurs dans le Banach V. Sur ces formes, on peut faire opérer la différentiation extérieure définie par Kerner [Ann. of Math. **34**, 156–172 (1933)]. On peut alors obtenir le résultat principal de ce travail: les groupes de cohomologie de X définis à partie des r-V-formes invariantes par le groupe général d'holonomie H sont isomorphes aux groupes de cohomologie de l'espace vectoriel tangent V_p définis à partir des r-V-formes invariantes par le groupe linéaire d'holonomie dH_p. On en déduit, dans le cas des espaces homogènes généraux, une extension du classique théorème d'Elie Cartan. *A. Lichnerowicz* (Paris).

Sasayama, Hiroyoshi. On the space of line-elements of fractional order with derived metric extensors. Tensor (N.S.) **4**, 91–106 (1954).

This paper is concerned with further extensions to the results appearing in a paper by Katsurada [J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 17–28 (1951); MR **15**, 986]. Whereas the paper just referred to is concerned with the space of line element of integral order, the present author is concerned with spaces of line elements of fractional order. He develops the extended Riemannian geometry of fractional order μ by means of the derived metric extensors of fractional grade. Some features in which the theory for fractional order differs substantially from the theory for integral order are given. *E. T. Davies* (Southampton).

Sasayama, Hiroyoshi. On generalized spaces with metric of fractional order. Tensor (N.S.) **4**, 107–121 (1954).

The author is concerned with the construction of a "fractional" differential geometry. In this paper he generalizes what have been known as Kawaguchi spaces for 20 years, in which the fundamental function is given as a scalar function of the components of the line elements of an integral order m. In recent papers the author has defined components of line-elements of fractional order μ, and in this paper he studies spaces in which the arc length of a curve is given by the integral of a scalar function of the curve-parameter t and the components of the line-element of fractional order μ.

In the corresponding treatment of the variational problem of fractional order, there are two kinds of Euler vectors and three kinds of Synge vectors. The author later develops a theory of affine connection in the space. *E. T. Davies.*

Tonooka, Keinosuke. On intrinsic theories in the manifold of surface-elements of higher order. II. Intrinsic geometry of a system of partial differential equations. Tensor (N.S.) **4**, 67–77 (1954).

The invariant theory of a system of partial differential equations

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} = H^i{}_{\alpha\beta}\left(u^\gamma, x^j, \frac{\partial x^j}{\partial u^\gamma}\right) \tag{1}$$

under the combined group of point and parameter transformations

$$x^{i'} = x^{i'}(x^j), \quad u^{\alpha'} = u^{\alpha'}(u^\beta) \tag{2}$$

was initiated by Bortolotti [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **23**, 16–21, 104–110, 175–180 (1936)] and called by him the intrinsic geometry of the system of equations.

In this paper the author considers a two-fold generalization. In the first part of the paper the second-order differential equations (1) are replaced by equations of arbitrary order m (>2), and in the second part the group of transformations (2) is replaced by the so-called generalized rheonomic group $x^{i'} = x^{i'}(x^\beta, u^j)$, $u^{\alpha'} = u^{\alpha'}(u^\beta)$. The two intrinsic operators by the author in a previous paper [J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 43–72 (1952); MR **14**, 1017] are again used extensively here. Some of the main theorems proved in the first part devoted to the study of equations under group (2) are shown to hold when the group of transformations is (3). *E. T. Davies* (Southampton).

Tandai, Kwoichi. On areal spaces. VII. The theory of the canonical connection and m-dimensional subspaces. Tensor (N.S.) **4**, 78–90 (1954).

The author in a recent paper [Tensor (N.S.) **3**, 40–45 (1953); MR **15**, 254] has proved that the only areal spaces for which a unique and real-valued 2-index metric tensor can be deduced from a fundamental integrand function are the spaces of Riemann, Finsler and Cartan. In this paper he proves that for areal spaces in general it is still possible to define a class of metric tensors which he indicates by $\{g_{ij}\}$. With the help of this class of metric tensors he constructs a 4-index metric tensor depending only on the class, and shows how this 4-index tensor may be used to define the length of a contravariant vector and the angle between two contravariant vectors. Three kinds of connection are then introduced, the first two depending on individual tensors g_{ij} of the class $\{g_{ij}\}$ while the third depends only on the class. This is derived from the first two and is called the canonical connection. With this canonical connection the vectors of curvature of the space is defined. The paper ends with the theory of subspaces in areal spaces so defined. *E. T. Davies* (Southampton).

Ōtsuki, Tominosuke. Theory of connections and a theorem of E. Cartan on holonomy groups. I. Math. J. Okayama Univ. **4**, 21–38 (1954).

If in a fiber bundle $\mathfrak{B}$ an infinitesimal connection Γ in the sense of Ehresmann is given (i.e. there is defined a "transversal field" of n-elements in the bundle space which by the projection p of the bundle are mapped onto the tangent spaces of the base space, and which associate with each path C in the base space X an isomorphism $\rho(C)$: $Y_x \to Y_y$, where x and y are the initial and end points of C), then, as Ehresmann proved, the given bundle $\mathfrak{B} = (B, p, X, Y, G)$ is G-equivalent to a bundle $\mathfrak{B}' = (B', p', X, Y, H)$, where H is

the holonomy group of the connection Γ. The present author proves the following theorem, which is an extension of a local theorem by E. Cartan: Let $\mathfrak{B}$ be a differentiable fiber bundle with an infinitesimal connection Γ whose structure group is G. Let H be the holonomy group of Γ at a point $x \in X$, and let K be the minimal invariant subgroup of G which contains H. Then $\mathfrak{B}$ with Γ is G-equivalent to another fiber bundle $\mathfrak{B}'$ with an infinitesimal connection Γ' whose structure group is L, where 1) if X is an n-cell, then $L=H$; 2) if X is simply connected, then $L=K$; 3) otherwise $L=G$.

The author takes for the definition of H not the same group that Ehresmann takes, but its closure in G. There doesn't seem to be any particular reason why this choice was made. In view of a result by Ambrose and Singer on the Lie algebra of the holonomy group [Trans. Amer. Math. Soc. **75**, 428–443 (1953); MR **16**, 172] Ehresmann's choice seems to be the preferable one. [The author's remark that every path of class D^1 in the base space can be lifted up in the large into a horizontal curve in B as a consequence of the assumptions 1–3 of §1 is true if one assumes that in the principal bundle the connection is right-invariant, but not in general.] Finally, the definition of "structure group of the connection" is not very clear, and the subsequent context does not seem to clarify the concept entirely, either.

A. Nijenhuis (Princeton, N. J.).

Berthé, Guy. Sur la symétrie sphérique d'un tenseur de rang 2. C. R. Acad. Sci. Paris **240**, 1313–1314 (1955).

The author claims, wrongly, to derive expressions for the generators of the group of rotations of a spherically symmetric space-time

$$ds^2=\alpha(r)dr^2+\beta(r)(d\theta^2+\sin^2\theta\,d\phi^2)+\gamma(r)dt^2$$

more general than the usual ones (e.g., as given by Vaidya [Phys. Rev. (2) **90**, 695–698 (1953), equations (2.5) and (2.6); MR **15**, 168] in discussing spherically symmetric skew tensors).

The generators cannot have the form $\xi^r=\beta^{-2}\xi^{r*}(\theta,\phi)$ which is found (ξ^{r*} certain functions of θ and ϕ alone). Such a generator generates not a motion but a sort of shearing rotation which does not satisfy Killing's equations, as asserted. Vaidya's generators are the correct ones. However, the most general spherically symmetric skew tensor is not Vaidya's, referred to above, nor his corrected version [ibid. (2) **96**, 5–9 (1954); MR **16**, 184], but that given previously by Papapetrou [Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948); MR **10**, 580]. The forms given by Vaidya are not invariant under general rotations [this modifies the reviewer's earlier remarks in MR **16**, 184; cf. the review by Schild, MR **15**, 168]. Consequently, the last three sentences of the paper under review are misleading.

F. A. E. Pirani (Dublin).

NUMERICAL AND GRAPHICAL METHODS

***Booth, Andrew D. Numerical methods.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1955. vii+195 pp. $6.00.

"The present book has grown out of the series of lectures given by the author to final honours B.Sc. mathematics students at Birkbeck College, London. The purpose of the course has been, not so much to instruct in the detailed tedium of actual computation, but rather to give an understanding of the basic principles upon which such analyses rest."

The author then covers the following topics: The nature and purpose of numerical analysis (6 pp.); tabulations and differences (5 pp.); interpolation (16 pp.); numerical differentiation and integration (23 pp.); the summation of series (5 pp.); the solution of ordinary differential equations (16 pp.); simultaneous linear equations (38 pp.); partial differential equations (32 pp.); non-linear algebraic equations (19 pp.); approximating functions (9 pp.); Fourier synthesis and analysis (10 pp.); integral equations (9 pp.).

The author covers so wide a range of subject matter for such a short work by minimizing the use of illustrative examples and by eliminating mathematical derivations which would help "to give an understanding of the basic principles upon which such analyses rest". For example, the error produced by Simpson's rule is incorrectly given on page 49. The author's attempt to present an uncluttered and useful account of numerical methods suitable for large scale computors will have to be revised and amplified if he wishes to fulfill the aim quoted from the preface: "The classical methods of hand calculation are, to a greater or less extent, unsuitable for the modern machines, and only by having a thorough knowledge of the underlying mathematical principles, is the programmer likely to make effective use of the new tools".

E. Isaacson.

Zondek, B. The values of $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ and their logarithms accurate to 28 decimals. Math. Tables Aids Comput. **9**, 24–25 (1955).

***Savage, Richard, and Lukacs, Eugene. Tables of inverses of finite segments of the Hilbert matrix.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 105–108. National Bureau of Standards Applied Mathematics Series No. 39, U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

The Hilbert matrix is the infinite matrix with elements $a_{ij}=1/(i+j-1)$. The paper gives exact inverses, in fractional form, of the $n\times n$ segments for which $i,\ j\leq n$, $n=2(1)10$. The result demonstrates the high degree of ill-conditioning.

J. C. P. Miller (Cambridge, England).

***Todd, John. The condition of the finite segments of the Hilbert matrix.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 109–116. National Bureau of Standards Applied Mathematics Series No. 39, U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

Segments of the Hilbert matrix, mentioned in the previous review, are notoriously ill-conditioned with respect to inversion. This paper studies various criteria of ill-conditioning, such as Turing's M-condition number, in connection with the 4×4, 5×5 and 6×6 segments of the Hilbert matrix. Various experimental inversions were carried out numerically on SEAC, to study the effects of rounding and other errors in cases of known ill-conditioning. Some failed completely, the partial success of others is exhibited.

J. C. P. Miller (Cambridge, England).

Horvay, G., and Born, J. S. Tables of self-equilibrating functions. J. Math. Phys. **33**, 360–373 (1955).

In connection with problems of finding stress distributions in semi-infinite strips due to prescribed self-equilibrating normal tractions or to prescribed self-equilibrating shear tractions, applied at the left edge, Horvay [J. Appl. Mech. **20**, 87–94 (1953); MR **14**, 819] has developed approximate solutions. The present paper contains tables to help in the solution of problems based on his approximations.

The main table III gives five-decimal values of $f_n(y)$, $f_n'(y)$, $f_n''(y)$ and $\int_0^y f_n(t)dt$ for $n=2(1)9$, $y=0(0.5)1$, where $f_n(y)$ constitute an orthonormal set of polynomials satisfying the differential equation

$$(1-y^2)^2 f^{iv}-10y(1-y^2)f'''+(\lambda-24)(1-y^2)f''+(12-4\lambda)yf'-2\lambda f=0$$

with $\lambda=(n+2)(n+3)$. The polynomials and normalising constants are listed in Table IA. y is a variable across the strip normalised to the range $(-1, 1)$.

The other large Table IV, gives functions varying in the x-direction, along the strip. The functions tabulated are

$$g_n(x),\quad g_n'(x),\quad \int_0^x g_n(t),\quad h_n(x),\quad h_n'(x),\quad \int_0^x h_n(t)$$

to six decimals, with g_n'' and $h_n''(x)$ to 5 decimals, all for $n=2(1)9$, $x=0$, 0.1, $0.2(0.2)L$, L being 4 for $n=2$, and smaller (since values become negligible) for $n>2$. $g_2^{iv}(x)$ is also given to 4 decimals. The definitions of these functions are:

$$g_n(x)=e^{\alpha_n x}\left(\cos\beta_n x+\frac{\alpha_n}{\beta_n}\sin\beta_n x\right),\quad h_n(x)=e^{-\alpha_n x}\frac{\sin\beta_n x}{\beta_n}$$

with

$$\alpha_n+i\beta_n=(C^2+i(D^2-C^2)^{\frac{1}{2}})^{\frac{1}{2}},$$

$$C=\int_{-1}^{1} f_n'^2(t)dt,\quad D=\int_{-1}^{1} f_n''^2(t)dt.$$

Graphs of the various functions are also given.

J. C. P. Miller (Cambridge, England).

Uhlir, Arthur, Jr. The potentials of infinite systems of sources and numerical solutions of problems in semiconductor engineering. Bell System Tech. J. **34**, 105–128 (1955).

Author's summary: "Tables and charts are given of mathematical functions related to the potential of a line of point charges. The use of these functions is illustrated by applications to semi-conductor resistivity measurements and to calculations of the base resistance of point-contact transistors".

In detail, the main tables give six-decimal values for $\lambda=0.01(0.01)2(0.1)5$ of the functions

$$M(\lambda)=2\sum_1^\infty(n^{-1}-(n^2+\lambda^2)^{-1/2}),\quad N(\lambda)=4\sum_1^\infty K_0(2\pi n\lambda),$$

K_0 being the Bessel function, and five-decimal values of $M'(\lambda)$ for $\lambda=0.05(.05)2.5$ and of $-N'(\lambda)$ for $\lambda=1(0.05)2$. It is also noted that

$$M(\lambda)+N(\lambda)=\frac{1}{\lambda}+2\ln\lambda-2\ln 2+2\gamma,$$

where γ is Euler's constant, 0.577215665. Charts are also given allowing 2–3 figures to be read directly.

J. C. P. Miller (Cambridge, England).

Lauffer, R. Interpolation mehrfacher Integrale. Arch. Math. **6**, 159–164 (1955).

The integral of an arbitrary polynomial $P_m(x_1, \cdots, x_n)$ of degree m over the n-simplex $\sum x_i\leqq 1, x_i\geqq 0, i=1, 2, \cdots, n$, is expressed as a linear combination of values of P_m at an equally-spaced, rectangular set of lattice points. Values of the coefficients in these linear combinations are tabulated as functions of n for $m=1(1)5$. For $m=1$ the result is a generalization to n dimensions of the trapezoidal rule, for $m=2$ of Simpson's rule, etc. *P. W. Ketchum.*

Vernotte, Pierre. Application d'un même principe de régularité au dépouillement des courbes expérimentales et à la solution de problèmes d'analyse mathématique. Rev. Gen. Sci. Pures Appl. **62**, 11–13 (1955).

Mittmann, O. M. J. Zur praktischen Analyse von Beobachtungsfolgen oder empirischen Funktionen. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **8**, 113–120 (1955).

***Madwed, Albert. Numerical analysis by the number series transformation method.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 320–368. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

Let (f) denote the set $f(x_i)$ of the values of the function $f(x)$ for a discrete set of abscissae x_i. As is known, numerical methods, especially those of differences, require and supply information on the functions $f(x)$ in the form of sets (f). This set can be used to set up a polygon function, representing the linear interpolation between adjacent abscissae x_i. This polygon function approximates $f(x)$. Another approximation to $f(x)$ can be made by a stepwise constant function from the information given by (f). The approximations can be linearly combined from elementary functions, such as step functions, ramp functions or triangular functions.

Information from (f) is what the author calls the number series method. He discusses the approximation of f by polygon functions for the case of equidistant abscissae $x_i=ih$. Special emphasis is on the linear combination of polygon functions from the triangular functions $p_i=h^{-1}-h^{-2}|x-x_i|$ for $|x-x_i|\leqq h$ and $p_i=0$ otherwise. For p_i the symbol $[UTPF](x-i\cdot(\Delta x))$ is used. A more extensive symbolism of this kind is used to introduce operations on sets (f), (g) which run parallel to operations on $f(x)$ and $g(x)$, especially the elementary operations, differentiation, integration and convolution integrals. A numerical example refers to a linear and to a nonlinear differential equation. It is claimed that the method of operations on the sets (f), (g) opens a new field in instrument and control engineering.

H. Bückner (Schenectady, N. Y.).

Myškis, A. D., and Ègle, I. Yu. On an estimate of the error in the method of successive approximations. Mat. Sb. N.S. **35(77)**, 491–500 (1954). (Russian)

The authors give four essential characteristics of useful accuracy estimates for any approximate method: a) They should use only known quantities; b) they should not be excessively loose; c) they should be adaptable to a large number of ways of applying the basic method; d) they should not be too laborious to apply. They claim that few estimates satisfy all these criteria, but in this regard praise those of L. V. Kantorovič and V. I. Krylov, Approximate methods of higher analysis [e.g., 3rd ed., Gostehizdat,

Moscow-Leningrad, 1950; MR 13, 77]. The authors claim that for many estimates deficiencies of the above types can be corrected very easily. To illustrate this they consider the solution by successive approximation of the Volterra integral equation

$$y(x)=\phi(x)+\int_{x_0}^{x}F(x,s,y(s))ds,$$

where F and ϕ are continuous and

(*) $|F(x,s,y_1)-F(x,s,y_2)|\leqq K|y_1-y_2|$ (K=constant).

The algorithm treated is the usual one:

$$y_n(x)=\phi(x)+\int_{x_0}^{x}F(x,s,y_{n-1}(s))ds \quad (n=1,2,\cdots).$$

A common error estimate for $|y(x)-y_n(x)|$ is derived from (*), based on proving inductively that

$$|y_n(x)-y_{n-1}(x)|\leqq A(x-x_0)^p \quad (A\geqq 0, p>-1).$$

The authors observe that (*) does not ordinarily hold for all y in $(-\infty,\infty)$, and derive estimates under the weaker assumption that $K=K(a,b;x,s)$, for $a\leqq y\leqq b$. The estimates are then refined. There is an example corresponding to solving $y'=x^2+y^2$, $y(0)=0$.

In another section the authors generalize the procedure of getting a bound for $|y-y_n|$ from a bound for $|y_n-y_{n-1}|$ to mappings of a partially ordered abelian semigroup on itself. As an application, Volterra-type operators and equations in a Banach space are defined, and a Lipschitz condition analogous to (*) is used to give an error estimate for solutions by successive approximation. *G. E. Forsythe.*

Šmulyan, Yu. L. Remark on the paper of Yu. M. Gavrilov, "On convergence of iterative processes." Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 191 (1955). (Russian)
See same Izv. **18**, 87–94 (1954); MR **16**, 177.

Hestenes, Magnus R. Iterative computational methods. Comm. Pure Appl. Math. **8**, 85–95 (1955).

L'auteur passe en revue un certain nombre de méthodes qu'il a eu l'occasion d'expérimenter et qui se rattachent plus ou moins à la notion de gradient. Apres avoir défini cette notion dans divers cas (fonction, problème aux variations, équation intégrale) il rappelle brièvement la methode du gradient conjugué pour les systèmes linéaires. Il indique un procédé différent pour l'inversion d'une matrice. Enfin il donne quelques indicatrices sur la recherche des valeurs propres. *J. Kuntzmann* (Grenoble).

Rutishauser, Heinz. Anwendungen des Quotienten-Differenzen-Algorithmus. Z. Angew. Math. Phys. **5**, 496–508 (1954).

This is the practical sequel to the author's earlier theoretical paper on the quotient-difference (QD) algorithm [same Z. **5**, 233–251 (1954); MR **16**, 176]. Numerical examples are given of the following applications: (1) to convert the nowhere convergent series $-\sum_0^\infty n!(-z)^{-n}$ into a continued fraction convergent for all z not on the negative real axis; the power series for e^z is also treated; (2) to sum a slowly convergent numerical series; (3) to solve the algebraic equations $z^3-9z^2-8z+2=0$ and $z^4-4z^3+5z^2-4z+1=0$, including finding a real quadratic factor with complex roots. It is noted that the original QD algorithm for solving equations suffers badly from loss of precision, and a "progressive" form of the algorithm is given which is much more precise. This involves working downwards from the slanting top row of the QD-table. *G. E. Forsythe.*

Salzer, Herbert E. A simple method for summing certain slowly convergent series. J. Math. Phys. **33**, 356–359 (1955).

The author considers the nth partial sums of the given series as values of a function $f(x)$ by setting $x=1/n$, and then extrapolates this function to $x=0$ by means of a Lagrange polynomial through the m partial sums ending with the nth. Tables of the Lagrange coefficients are given for $n=5$, 10, 15, 20 and $m=4$, 7, 11. Examples are given of slowly convergent series of positive terms in which the method yields as much as 15-place accuracy from partial sums which differ from the actual sum by more than one unit. For certain alternating series the method fails when applied directly but succeeds when applied to alternate partial sums. *P. W. Ketchum* (Urbana, Ill.).

Traenkle, C. A. Determination of the root systems of algebraic equations by affinity transforms. Quart. Appl. Math. **13**, 1–18 (1955).

La méthode proposée pour trouver les facteurs x^2+px+q d'une équation algébrique à coefficients réels est une méthode de division. Si le reste est désigné par b_1x+b_0, il s'agit dans le plan des b_0b_1 d'atteindre l'origine. On opère d'abord par tatonnements, ensuite en admettant que p et q dépendent linéairement de b_0b_1. De nombreux détails sont discutés, entre autres les particularités relatives aux racines multiples. *J. Kuntzmann* (Grenoble).

Householder, A. S. On solving linear algebraic systems. Oak Ridge National Laboratory, Rep. **ORNL 1785**, 20 pp. (1954).

Some known and new terminating iterative methods for solving a linear system $Ax=y$ are given a uniform presentation, together with estimates of the effects of round-off error in a digital computer. The new ones are called methods of "orthogonal directions" and of "orthogonal gradients," being generalizations of related methods of M. R. Hestenes and E. Stiefel [J. Res. Nat. Bur. Standards **49**, 409–436 (1953); MR **15**, 651]. Let x_p be any digital vector. One has v and $u=Av$ in any of the above algorithms. One selects $x_{p+1}=x_p+\lambda v$ so that $r_{p+1}=y-Ax_{p+1}$ is orthogonal to u. Actually one forms the digital vector $x^*_{p+1}=x_p+(\lambda^* v)^*$. The principal round-off question is: when is $N(r_{p+1})<N(r_p)$, where $N(w)=(w^Tw)^{1/2}$? The answer is stated in terms of u, r_p, r_p^*, the order n, the elementary round-off error ϵ, and the upper bound of A. *G. E. Forsythe.*

Kostarčuk, V. N. On a method of solution of systems of linear equations and of finding the eigenvectors of a matrix. Dokl. Akad. Nauk SSSR (N.S.) **98**, 531–534 (1954). (Russian)

For the problems of the title the author proposes and discusses a variant of the method of steepest descent of Kantorovič [Uspehi Mat. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948); MR **10**, 380; **14**, 766]. Given the linear system $Bx=b$, where B is a positive definite matrix. Then $F(x)=(Bx-b, x-B^{-1}b)=c$ ($c\geqq 0$) defines a family of similar ellipsoids with $c=0$ at the center $x^*=B^{-1}b$. Let x_0 be arbitrary. In the proposed iteration one takes $x_{n+1}=x_n-2\alpha_n\Delta_n$, where $\Delta_n=Bx_n-b$, and $\alpha_n=(\Delta_n,\Delta_n)/(B\Delta_n,\Delta_n)$. Then $F(x_n)=F(x_0)$, and all points x_n lie on the same ellipsoid. By straightforward arguments the author proves for ordinary x_0, as $n\to\infty$, that $\Delta_{2n}\to e_M$, that $\Delta_{2n+1}\to -e_M$, and that $\frac{1}{2}(x_{2n}+x_{2n+1})\to x^*$, where e_M is an eigenvector of B belonging to its maximum eigenvalue. The speed of convergence is estimated. *G. E. Forsythe* (New York, N. Y.).

Benthem, J. P. Note on minimizing a quadratic function with additional linear conditions by matrix methods, with application to stress analysis. Nationaal Luchtvaartlaboratorium, Amsterdam. Report S. 437, i+8 pp. (1954).

Let A be an m-by-m matrix, while X and B are m-by-1 columns. Let C be a p-by-p matrix ($p<m$), while D is p-by-$(m-p)$. Let X_1, X_2 be p-by-1, $(m-p)$-by-1 columns, respectively, such that the partitioned matrix

$$\|(X_1)'(X_2)'\|'=X$$

(prime denotes transpose). Finally, E is a p-by-1 column. The mathematical problem is to find X minimizing $\frac{1}{2}X'AX+X'B$, subject to the linear constraints $CX_1=DX_2+E$. The answer is shown to be

$$(*) \qquad X=-Z(Z'AZ)^{-1}Z'(AY+B)+Y,$$

where $Z=\|(C^{-1}D)(I)\|'$, $Y=\|(C^{-1}E)(0)\|'$. Numerical evaluation of (*) involves inverting a matrix of order p and one of order $m-p$. In contrast, defining $F=\|(C)(-D)\|$, the method of Lagrange multipliers is shown to lead to the expression

$$X=A^{-1}\{F'(FA^{-1}F')^{-1}(E+FA^{-1}B)-B\},$$

whose evaluation entails matrix inversions of orders p and m. The physical problem is the solution of a statically indeterminate structure, and there is some discussion of its special character. *G. E. Forsythe* (Los Angeles, Calif.).

Kron, Gabriel. Detailed example of interconnecting piecewise solutions. J. Franklin Inst. 259, 307–333 (1955).

An example illustrates the author's technique of utilizing the physical setting (electrical networks) represented by a matrix as a guide in decomposing it for blockwise inversion. (Since a general, compact, universal idiom exists, it is a pity the author prefers the prolixity of his own.)

A. S. Householder (Oak Ridge, Tenn.).

***Milne, William Edmund. Numerical solution of differential equations.** John Wiley & Sons, Inc., New York; Chapman and Hall, Limited, London, 1953. xii+275 pp. $6.50.

The appearance of a book on the numerical solution of differential equations, written by an acknowledged expert in this field, is a major event in the computing world. The avowed aim of this book is to present in an elementary manner, and mainly for the benefit of the practical worker, some of the principal techniques available for solving ordinary and partial differential equations.

The introductory chapter shows the need for numerical methods, but asks the computer to use all his mathematical knowledge and gently chides the pure mathematician for his lack of interest in this subject.

Chapters 2 to 5 are concerned with the solution of a general ordinary differential equation of first order. Some nine methods, with variations, are described in considerable detail. The basis of most of the methods is the use of "predictor" formulae to approximate to a new pivotal value in the step-by-step process, followed by more accurate "corrector" formulae to improve this estimate. These formulae of integration may be of Lagrange type, or involve backward or central differences. Special procedures are described for starting the integration. The Runge-Kutta methods, and the use of Taylor's series in various contexts, are also included. Rounding errors are deliberately ignored, but the effect and estimation of truncation errors are examined in detail.

Chapter 6 considers systems of first-order equations and equations of higher order. Special predictor and corrector formulae are developed for second-order equations which lack the first derivative. The central-difference "summation" method is described, and for linear equations there is a brief mention of the Fox-Goodwin "difference-correction" process.

Two-point boundary problems are treated in Chapter 7. Methods suggested include combinations of step-by-step solutions to satisfy the boundary conditions, iteration based on the scheme $y''_{n+1}=f(x, y_n, y_n')$, and the methods of Ritz and Galerkin.

The rest of the book is concerned mainly with partial differential equations. Explicit methods, in which values are obtained directly in a step-by-step process, are used in Chapter 8 to solve parabolic and hyperbolic equations in one and two space dimensions. The latter problem introduces the Laplace operator in Cartesian coordinates, for which various finite-difference formulae and their error terms are derived.

Chapter 9, an introduction to implicit methods used in Chapter 10 to solve elliptic equations, gives a summary of the properties of matrices and of methods, mainly iterative, for solving linear algebraic equations and calculating the latent roots and vectors of matrices. Chapter 10 is concerned chiefly with Laplace's equation in x and y, with particular emphasis on two different forms of the corresponding finite-difference equations. Some properties of the resulting special matrices are established and methods of iteration, including relaxation with some useful group-relaxation patterns, are used to effect a solution. There is a brief analysis of error, a comment on the effect of singularities on the boundary, and a mention of the biharmonic equation.

Chapter 11 considers the calculation of characteristic numbers associated with ordinary and partial differential equations. This material is supplementary to that of Chapter 9, but methods based on the introduction of a "fictitious" vibration problem generally give a more accurate determination of the characteristic numbers than is provided by the matrix obtained from the ordinary finite-difference equations.

Brief notes on rounding errors, large-scale computing machines, and the Monte-Carlo method appear as Appendices, and there is a bibliography of no fewer than 260 entries. The text is illustrated throughout with worked examples, and exercises are provided except, strangely, in Chapters 6, 7 and 11, for the further practice of the reader.

A book of 280 pages cannot give comprehensive treatment to such a large subject, and some omissions are inevitable. The author has concentrated on those methods "deemed most essential", giving explicit reference in relevant places to other methods and variations. In the first 80 pages, on first-order equations, this plan has succeeded admirably. In Chapter 6, however, on higher-order equations, reference might have been made to the Clenshaw-Olver "η-correction" method [Math. Tables Aids Comput. 5, 34–39 (1951), not given in the bibliography], and Chapter 7, on two-point boundary problems, seems too short and rather pessimistic. For boundary-value problems in both ordinary and partial equations there is little mention of direct methods of solving the simultaneous equations, though the special forms of the corresponding matrices make these methods relatively easy: only the simplest equations and boundary conditions, the

latter nowhere including a derivative, are considered. L. F. Richardson's method of "the deferred approach to the limit", applicable in many problems, is excluded. The most serious omission, however, is that of the "method of characteristics", almost essential for solving non-linear hyperbolic equations. Chapters 9 and 11, full of valuable material, suffer in clarity and ease of reading through inevitable compression.

Despite this, and the presence of several misprints and one or two minor factual errors, this book is a mine of information and will repay careful study by all, computers and mathematicians alike, who are interested in differential equations. The printing is excellent. *L. Fox.*

***Richter, Willy. Estimation de l'erreur commise dans la méthode de M. W. E. Milne pour l'intégration d'un système de n équations différentielles du premier ordre.** Thèse, Université de Neuchâtel, 1952. 43 pp.

This thesis gives a thorough investigation of maximum errors that may occur during the use of Milne's method [Numerical calculus, Princeton, 1949, p. 135; MR **10**, 483] for step-by-step solutions of several simultaneous first-order differential equations from initial conditions given at one point. The method involves initial calculation of values at four points (the initial point and three others) by an iterative procedure, followed by a familiar step-by-step procedure of first "predicting" and then "correcting" values at the end of the next step, by use of appropriate formulae. The convergence of the iterative procedures, and the maximum errors possible for each of the processes used are investigated in detail, subject to conditions assumed for the functions involved. The results are then tested by application to an example of some antiquity, namely the equation $dy/dx=x^3+y^3$. *J. C. P. Miller.*

Gautschi, Walter. Über die zeichnerischen Ungenauigkeiten und die zweckmässige Bemessung der Schrittlänge beim graphischen Integrationsverfahren von Meissner-Ludwig. Verh. Naturf. Ges. Basel **65**, 49–66 (1954).

The first-order integration procedure of Meissner-Ludwig consists of a step-by-step graphical procedure for solving the equation $y'=f(x, y)$ such that $y_{n+1}=y_n \cos h+y_n' \sin h$, $y_n'=f(x_n, y_n)$. The author analyses the graphical procedure and obtains over-estimates for the inaccuracy of this process at each step. He indicates methods for comparing the error due to the use of a finite size step and the inaccuracy due to the graphical procedure. It seems clear that for a given interval and given differential equation system with reasonable properties a step size exists for which these are equal. The author seems to suggest ways of finding this step size, but the method is not clear to the reviewer.

F. J. Murray (New York, N. Y.).

Lotkin, Mark. The improvement of accuracy in integration. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. No. **912**, 25 pp. (1954).

In integrating numerically ordinary differential equations one commonly makes at least two computations of the dependent variable at each new point, whether by "predictor" and "corrector" formulas or otherwise. The author combines the estimates for the two errors (including truncation and roundoff), along with the observed difference between the two results, to obtain an improved estimate for the error in the computation considered to be the more exact. This is used in the first place to yield a third and final evaluation, and, in the second place, to determine whether or not the interval length may be increased or should be decreased. Application is made to particular methods (e.g., Heun's), and a numerical example is worked out.

A. S. Householder (Oak Ridge, Tenn.).

Lotkin, Mark. On the improvement of accuracy in integration. Quart. Appl. Math. **13**, 47–54 (1955).

Consider a system of n ordinary differential equations of the first order. Then the following improvements are considered. (1) The elimination of the major error in two approximations to the values of y at the next point of integration. The two estimates may be found either by a "predictor-corrector" technique, or by repeated application of the same formula at different intervals. (2) The use of the two approximations to increase or decrease the interval, in order to keep the error with a prescribed limit.

The method is illustrated by the numerical integration of $y'=y^3$, using the formulae of Euler-Heun, viz: $y_{k+1}=y_k+hy_k'$ as estimator, and $y_{k+1}=y_k+\frac{1}{2}h(y_k'+y_{k+1}')$ as corrector.

D. C. Gilles (Manchester).

Urabe, Minoru, and Yanagihara, Hiroki. On numerical integration of the differential equation $y^{(n)}=f(x, y)$. J. Sci. Hiroshima Univ. Ser. A. **18**, 55–76 (1954).

This paper contains results of further investigations, for ordinary differential equations of second and third orders along the lines of an earlier paper by Urabe and Tsushima [same J. **17**, 193–219 (1953); MR **16**, 290].

J. C. P. Miller (Cambridge, England).

Oshida, Isao. A new graphical solution of ordinary linear differential equation of the second order. J. Phys. Soc. Japan **10**, 228–232 (1955).

La solution approchée de l'équation $y''+f(x)y=0$ par la formule:

$$y_{i+1}=y_i+hz_{i+1/2} \quad z_{i+1/2}=z_{i-1/2}-hf(x_i)y_i$$

peut se mettre sous la forme d'une construction assez semblable à un abaque. *J. Kuntzmann* (Grenoble).

***Picone, Mauro. Exposition d'une méthode d'intégration numérique des systèmes d'équations linéaires aux dérivées partielles, mise en oeuvre à l'Institut National pour les Applications du Calcul. Résultats obtenus et résultats que l'on pourrait atteindre.** Les machines à calculer et la pensée humaine, pp. 239–261. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

The author reports on work done at his institute in developing a method for the numerical solution of the non-homogeneous biharmonic equation, which arises in elasticity, $\Delta\Delta u=\mu f(x, y)$, in some domain D, for appropriate boundary conditions. Practical engineering requirements may call for information on the value of the bending moment $M(u)$, and the shear force $E(u)$ at the boundary of D in the case where the plate is clamped, i.e. $u=\partial u/\partial n=0$ along the boundary. ($M(u)$ involves second derivatives of u, while $E(u)$ contains third derivatives of u.)

He indicates how, for a simply connected domain D, these quantities $M(u)$ and $E(u)$ may be expressed as linear combinations with undetermined coefficients γ_i of the "homogeneous biharmonic polynomials" $\{v_i(x, y)\}$. That is, integration by parts is applied to reduce the integral identity

$$\iint_D (u\Delta\Delta v_i-v_i\Delta\Delta u)dxdy=-\mu\iint_D v_i f\,dxdy$$

to a linear equation in the undetermined coefficients γ_i. It can be shown that the finite set of linear equations in the $\gamma_i^{(\nu)}$, obtained by using the first ν polynomials $v_1, \cdots, v_\nu$, is uniquely solvable. For the case of a square plate, graphs are given to show the accuracies obtained by using polynomials of degree 4, 6, 8 and 10. It is remarkable that good agreement with the boundary-values of second and third derivatives is obtained from such low order approximations.

The author describes the variations necessary to handle problems for multiply connected domains and for elasticity problems in three independent space variables. He indicates the need for large high-speed computing machines in the solution of these problems. *E. Isaacson.*

Vaisey, Gillian, and Hemp, W. S. Solution of a load diffusion problem by relaxation methods. Coll. Aero. Cranfield. Note no. 17, 13 pp. (6 plates) (1954).

Lotkin, Mark. The solution by iteration of nonlinear integral equations. J. Math. Phys. 33, 346–355 (1955).

The author considers the integral equation $y(x)=Ly(x)$ with the operator

$$Ly=\sum_{i=1}^{r}\int_a^b\cdots\int_a^b K_i(x,s_1,\cdots,s_i)\times F_i(s_1,\cdots,s_i;u_1,\cdots,u_i)\,ds_1\cdots ds_i,$$

where $u_n=y(s_n)$. Assumptions are: All functions defined for $a\leqq x\leqq b$, $a\leqq s_n\leqq b$, $-c\leqq u_n\leqq c$; the functions F_i are continuous, the kernels K_i are integrable and admit integrable squares with respect to the variables s_n, the integrated squares are continuous in x; Lipschitz-conditions

$$|F_i(s_1,\cdots;u_1,\cdots,u_i)-F_i(s_1,\cdots;v_1,\cdots,v_i)|\leqq f_i\sum|u_n-v_n|$$

are satisfied with constants f_i independent of the variables s_n. A tacit assumption seems to be that a continuous $y(x)$ is transformed into a continuous Ly. The iteration method $y_{m+1}=Ly_m$ is studied. The author shows, that convergence is obtained under certain conditions and that the solution is unique. For $r=2$, the condition is $0\leqq E/(1-2D)\leqq c$ with

$$E=(b-a)^\alpha\cdot KM,\quad D=(b-a)^\alpha\cdot Kf,\quad f^2=f_1^2+f_2^2,$$
$$M^2=\max F_1^2(s,0)+\max F_2^2(s,t,0,0);$$
$$K^2=\left\|\int K_1^2ds+\int K_2^2dsdt\right\|;$$

$\alpha=1$ for $b-a\leqq 1$ and $\alpha=2$ for $b-a>1$. One of these conditions refers to the norm $\|g(x)\|=\max|g(x)|$ of a function g, which is continuous for $a\leqq x\leqq b$. Error estimates are given and a numerical example illustrates the method.

The reviewer believes that author's results are widely independent of the type of the integral equation. Let L be any operator, transforming the continuous functions of the interval $a\leqq x\leqq b$, which satisfy $|y(x)|\leqq c$ into continuous functions of the same interval. Then with $f(x)=Ly_0$, $y_0\equiv 0$, $L^*y=Ly-f(x)$, the problem is to solve $y=L^*y+f$. Then with $\|f\|\leqq E$, $\|L^*y\|\leqq 2D\|y\|$, D independent of y, $2D<1$, the author's condition ensures that $\|Ly\|\leqq c$ for $\|y\|\leqq c$. The iteration $z_{m+1}=Lz_m$ is possible if $\|z_0\|\leqq c$. Convergence to the unique solution follows from $2D<1$ due to the abstract fixed-point theorem of J. Weissinger [Math. Nachr. 8, 193–212 (1952); MR 14, 478]. *H. Bückner.*

Roesler, F. C. Some applications of Fourier series in the numerical treatment of linear behaviour. Proc. Phys. Soc. Sect. B. 68, 89–96 (1955).

In the one-dimensional theory of Maxwellian viscoelasticity the behavior of materials is characterized by a relaxation spectrum. If certain other functions (dynamic elasticity, relaxation function) are given, the spectrum can be determined by solving certain integral equations. These can be transformed into integral equations of the convolution type:

$$2E'(\omega)=\int_{-\infty}^{+\infty}[1+\tanh(\tau-\omega)]\phi(\tau)d\tau,$$
$$\sigma(\tau)=\int_{-\infty}^{+\infty}\exp[-\exp(\tau-\omega)]\phi(\omega)d\omega.$$

The author develops a method of solving these equations numerically. It consists in approximating all occurring functions by Fourier expansions in a sufficiently large interval. *W. Noll* (Berlin).

Weil, Herschel. Reduction of runs in multiparameter computations. J. Assoc. Comput. Mach. 2, 99–110 (1955).

***Kobrinskiĭ, N. E. Matematičeskie mašiny nepreryvnogo deĭstviya. Osnovy ih ustroĭstva. [Mathematical machines with continuous action. Fundamentals of their construction.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 447 pp. 16.05 rubles.

This book describes for the most part well-known and generally-accepted forms of mechanical, electromechanical, electrical network and electronic devices for the solution of algebraic, ordinary and partial differential equations, for summation, multiplication, division, integration and differentiation of variables, and for generation of special and arbitrary functions.

Selected for a particularly detailed discussion, accompanied by photographs, were finite-difference networks for solving partial differential equations and a 6-integer mechanical differential analyzer. By American standards the apparatus described is of highly limited capacity and rather crude. Several pages are given to describing a hydraulic model consisting of standpipes interconnected by small tubes at their bases and used for the solution of Laplace, Poisson and Fourier equations in two-dimensional regimes. The hydraulic apparatus was described as having certain advantages over electrical network methods.

A notable feature of this book is its detailed discussion of errors in the various elements of computing devices, both electrical and mechanical, and the effects of these errors on the accuracy of solutions obtained. *W. W. Soroka.*

Herbst, Eugene H., Metropolis, N., and Wells, Mark B. Analysis of problem codes on the Maniac. Math. Tables Aids Comput. 9, 14–20 (1955).

Beyer, H. Die rechnerischen Vorteile der "Curta"-Rechenmaschine. Österreich. Ing.-Arch. 9, 31–37 (1955).

de Boer, J. The electronic automatic computer "MIRACLE". Nederl. Tijdschr. Natuurk. 21, 41–43 (1955). (Dutch)

Graney, Edward P. Maintenance and acceptance tests used on the MIDAC. J. Assoc. Comput. Mach. 2, 95–98 (1955).

Cohn, Harvey. Some experiments in ideal factorization on the MIDAC. J. Assoc. Comput. Mach. 2, 111–116 (1955).

Moshos, George J. Analog interpolation for automatic control. J. Assoc. Comput. Mach. **2**, 83–91 (1955).

Knappe, Werner. Hydrodynamisches Modell zur Darstellung von Diffusionsvorgängen mit konzentrationsabhängigen Diffusionskoeffizienten. Z. Angew. Math. Phys. **6**, 140–142 (1955).

It is possible to get approximate solutions of the differential equation for non-stationary diffusion (Fick's 2nd law) by means of a hydrodynamic model consisting of cylindric containers connected by capillary tubes.

From the author's summary.

Palmer, P. J., and Redshaw, S. C. Experiments with an electrical analogue for the extension and flexure of flat plates. Aero. Quart. **6**, 13–30 (1955).

Ryder, Frederick L. Linear algebraic computation by multi-winding transformers. J. Franklin Inst. **259**, 427–439 (1955).

ASTRONOMY

Evrard, Léon. Sur l'équilibre relatif des fluides hétérogènes en rotation. Le problème des étoiles doubles. Ann. Astrophys. **14**, 17–39 (1951).

Two stars rotating with equal angular velocities are revolving about each other with an angular velocity of revolution equal to that of the stars' rotation. Considering the second star's mass concentrated in a point the author derives the equilibrium shape of the first star using terms through the fourth power of the angular velocity. He finds a depression forming at the 45° latitude levels in what would otherwise be an ellipsoid elongated in the direction of the second star. *R. G. Langebartel* (Urbana, Ill.).

***Vernić, Radovan. Diskussion der Sundmanschen Lösung des Dreikörperproblems.** Jugoslavenska Akademija Znanosti i Umjetnosti, Zagreb, 1954. 146 pp. 1140 Din.

This memoir consists of two parts. Part 1. Historical and critical discussion (81 pp.). In this part the author is concerned (i) with singularities of collision in the restricted and the general problem of three bodies, resp. the n-body problem ($n \geqq 4$), and the problem of their regularization; (ii) with asymptotic behavior of integrals in the three-, resp., the n-body problem; (iii) with the same problems as in (i) and (ii) but with reference to a generalized three-, resp., n-body problem when the Newtonian law of force is replaced by some other types of laws, such as an inverse power law, an exponential law or a logarithmic law.

Part 2. Uniformization of the three-body problem (56 pp.). The starting point is a formulation of Sundman's results on collision and regularization from a unified point of view. All cases of the two- and the three-body problem are regularizable and integrable by the method and the transformation used by Sundman. The most general transformation of this type is

$$Cdt = S(r)du, \tag{1}$$

where $S(r) = S(r_1, r_2, r_3)$ is a symmetric homogeneous function of degree one in all three mutual distances (extended Sundman's theorem).

Consider u and t as complex variables (which are eventually restricted to be real). Then the results on collision in the three-body problem can be stated as follows. The general solution of the three-body problem has at most a denumerable infinity of algebraic branch points t_ν (points of collision) at which two or three sheets of the corresponding Riemann surface unite. The dates t_ν of collisions cannot cluster at a finite limiting t^*.

In order to analyse the structure of the general solution, one has (i) to uniformize simultaneously all the branch points t_ν and (ii) to study the transcendental functions which represent the general solution. For this purpose the author introduces the concept of an algebromorphic function, defined as an analytic (in general transcendental) multiple-valued function $w = w(t)$ which in any finite part of the t-plane has at most a finite number of algebraic branch points as critical points: $w = A(t; t_\nu)$. The algebromorphic functions are regular and single-valued in the Mittag-Leffler star obtained by cutting the t-plane radially from the points t_ν to infinity. The simplest algebromorphic functions, determined by their branch points t_ν, at which p_ν sheets of the Riemann surface unite and which are of degree n_ν ($\gtrless 0$), are called the fundamental algebromorphic functions. Such an algebromorphic function with branch points from the denumerable set $\{t_\nu\}$ without finite cluster points is represented in the Mittag-Leffler star by the fundamental formula (a generalization of Weierstrass's product form for transcendental integral functions)

$$w = A(t; t_\nu, n_\nu, p_\nu) = \prod_{\nu=0}^{\infty} \left[\left(1 - \frac{t}{t_\nu}\right) \exp q_\nu(t) \right]^{\alpha_\nu}, \tag{2}$$

$\alpha_\nu = n_\nu/p_\nu \gtrless 0$, $p = m\{p_\nu\} < P$, where p is the bounded smallest common multiple of all p_ν. This last assumption is made in order that the corresponding Riemann surface have a finite number of sheets. The convergence-producing factors have to be chosen in such a way that $\sum_{\nu=0}^{\infty} \alpha_\nu (t/t_\nu)^{k_\nu} = K(t)$ is absolutely convergent; for example, one may take

$$q_\nu(t) = \sum_{\lambda=1}^{k_\nu - 1} (1/\lambda)(t/t_\nu)^\lambda.$$

The most general algebromorphic function with given properties is then given by $w = E(t; A)$, where A is given by (2) and the integral function E does not introduce any other critical points than those contained in A (existence theorem of algebromorphic functions). After this preparatory work the fundamental theorem of the three-body problem is stated in terms of algebromorphic functions as follows. The general solution of the three-body problem with branch points t_ν (points of collision) is given by

$$r_i = e^{E_i(t; A_i)} A_i(t; t_\nu). \tag{3}$$

Further, it is shown that algebromorphic functions are binomial algebroids, i.e. $w = A(t)$ satisfies a binomial algebraic equation $w^p - E(t)M(t) = 0$, where $E(t)$ is an integral function and $M(t)$ is a meromorphic function.

In the following section the theory of algebromorphic functions is applied to the two-body problem, and it is shown that the algebromorphic integration of the parabolic and the elliptic case (as well as that of the hyperbolic case which is not carried out) gives the classical integrals.

The last section is concerned with uniformization of the three-body problem. Any integration of the two- and the three-body problem is a uniformization of the corresponding multiple-valued algebromorphic functions (fundamental

theorem of uniformization of the three-body problem). This theorem provides a deeper understanding of the regularization problem. In the general case of the three-body problem one must distinguish between uniformization of the trajectories in the parametric form

$$x_i=x_i(s),\quad y_i=y_i(s),\quad z_i=z_i(s)\quad (i=1,2,3)$$

and uniformization of the motion (r_i, t) in the parametric form

$$x_i=x_i(u),\ \cdots,\ r_i=P_i(u),\quad t=T(u).$$

Uniformization of the trajectories is a geometric uniformization of the problem, while that of the motion (r_i, t) is its kinematic uniformization. Any uniformization of the two- and the three-body problem is not necessarily a regularization of the problem concerned and hence it is also not an integration of the problem. In general it is necessary to uniformize, in addition to the coordinates and the mutual distances, also the time t by use of the Sundman transformation (1). Only by this last operation is a uniformization of the trajectories generalized to a uniformization of the motion. The function $S(r)$ is the integrating factor which makes Cdt in (1) an exact differential.

In the case of the two-body problem the geometric uniformization is at the same time its kinematic uniformization (for example, for the elliptic case the geometric and the kinematic uniformizing parameter u is the eccentric anomaly E, for the parabolic case it is $\theta=\tan\frac{1}{2}v$, where v is the true anomaly). This fact is the deepest reason for the complete integrability of the two-body problem.

The global uniformizing parameter u which by (1) regularizes the general three-body problem is an algebromorphic function, given by a generalized Kepler equation of the form

$$u=-2ht+\sum_{i=1}^{3}\frac{e^{2E_i}}{m_i}A_i^2\left[\frac{\partial E_i}{\partial t}+\frac{\partial E_i}{\partial A_i}\frac{dA_i}{dt}+\frac{d\ln A_i}{dt}\right],$$

the general integral being given by (3) (final theorem on uniformization of the three-body problem).

The bibliography of the literature cited contains 170 items and extends to 1950. A short review of the corresponding literature for the period 1950–1952 is appended.

E. Leimanis (Vancouver, B. C.).

Vernić, Radovan. Die Stossbedingungen im Dreikörperproblem. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II **9**, 3–13 (1954). (Serbo-Croatian summary)

This paper is a continuation of a paper in the same Glasnik **8**, 247–266 (1953) [MR **16**, 181] and of the book reviewed above. In Section I, beginning with some well-known results of Sundman and Chazy, the author clarifies some points on double and triple collision left open in those two papers. Section 2 is concerned with triple collisions (real and imaginary) in the Lagrange's exact solutions of the three-body problem. Finally, in section 3, starting with the transformed Lagrangian formula

$$\frac{d}{du}\left(V\frac{dJ}{du}\right)=2\left(1+\frac{2h}{V}\right),\tag{1}$$

conditions for double and triple collisions are given (Theorems 7 and 8). These theorems, however, contain superfluous conditions due to the fact that the solution of (1) contains the solution of the three-body problem. The truth of the two theorems is verified for the two-body problem.

E. Leimanis (Vancouver, B. C.).

Merman, G. A. The restricted parabolic problem of three bodies. Byull. Inst. Teoret. Astr. **5**, 606–616 (1954). (Russian)

Consider the motion of three particles P_0, P_1, P_2 subject to their mutual attractions according to the Newtonian law, except that P_0 of mass $m_0=0$ does not disturb the motion of the finite particles P_1 and P_2 whose relative orbit is a parabola. J. Chazy [Ann. Sci. Ecole Norm. Sup. (3) **39**, 29–130 (1922)] showed that in the general three-body problem for $h=0$ (h being the constant of energy) hyperbolic-elliptic motions (one of the three mutual distances remains bounded while the other two distances increase indefinitely as $t\to\infty$ and are of the same order of magnitude as t) or parabolic motions (all three mutual distances increase indefinitely as $t\to\infty$ and are of the order of magnitude $t^{2/3}$) are possible but parabolic-hyperbolic and parabolic-elliptic motions are impossible. From this it follows that in the general three-body problem for $h=0$ the probability of a capture is zero.

Since on the one hand the restricted parabolic problem of three bodies is the limiting case of the general three-body problem for $h=0$ when one of the three masses approaches zero, a capture in this case seems impossible. On the other hand, however, the restricted parabolic problem can be considered as the limiting case of parabolic-hyperbolic (for $h>0$) or parabolic-elliptic motion (for $h<0$) as the mass of the body describing a hyperbola ($h>0$) or an ellipse ($h<0$) approaches zero. These two limiting cases show the possibility of occurrence of parabolic-hyperbolic or parabolic-elliptic motions in the restricted parabolic problem, these two types of motion being impossible in the general three-body problem for $h=0$. Hence capture in the restricted parabolic problem of three bodies in general is possible.

It is shown in the paper that all the types of motion mentioned above in the restricted parabolic problem are possible if certain criteria which are too detailed to be reproduced here are satisfied. The restricted parabolic problem has many features in common with the restricted hyperbolic problem considered previously by the author [same Byull. **5**, 325–372 (1953); MR **16**, 293].

E. Leimanis.

Kočina, N. G. An example of motion in the restricted parabolic problem of three bodies. Byull. Inst. Teoret. Astr. **5**, 617–622 (1954). (Russian)

A numerical example is given illustrating the occurrence of motion which is parabolic-elliptic in both directions of the time axis in the restricted parabolic problem of three bodies. The class of motion, however, has changed in the sense that for $t\to+\infty$ the orbit of P_0 relative to P_1 while for $t\to-\infty$ that of P_0 relative to P_2 is an osculating ellipse [see also the preceding review].

E. Leimanis.

Parilskiĭ, N. N. Computation of a trajectory of capture in the plane problem of three bodies of equal mass. Voprosy Kosmog. **3**, 33–62 (1954). (Russian)

The author gives a detailed account with eight numerical tables of the computation of the trajectory of capture carried out under his supervision in the example of O. Yu. Šmidt [Dokl. Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947)] for the time interval from $t=-8000$ to $t=129764$. Much attention is paid to the estimation of systematic errors committed by calculating the values of the coordinates and the velocities at the endpoints of the interval of numerical integration. Use is made of the energy integral and the integral of the moment of momentum of the three bodies.

The question as to whether the calculated trajectory corresponds to a capture in the problem under consideration is answered in the affirmative quite convincingly by an intuitive argument rather than by a mathematical one, but no strict mathematical proof for the occurrence of capture is given.

The author remarks that all his computations have been carried out as early as 1947 and the results partly published in Šmidt's paper with omission of the last two digits. However, the table reproduced by Šmidt contains a grave misprint at the end-point $t=129764$: instead of $x_1=7004$ it should read $x_1=17004$. This misprint has caused V. F. Proskurin [Byull. Inst. Teoret. Astr. **5**, 429–434 (1953); MR **16**, 294] and Merman [ibid. **5**, 325–372 (1953); MR **16**, 293] to claim that his computations are erroneous. On the other hand, in Hil'mi's book [The problem of n bodies in celestial mechanics and cosmogony, Izdat. Akad. Nauk SSSR., Moscow, 1951] misprints of his results are at the other endpoint $t=-8000$ (instead of $\dot{x}_1=0.08593, x_2=321.50$ it should read $\dot{x}_1=0.07172$ and $x_2=321.49$) and at the point $t=0$ (instead of $y_1=49.956$ should be $y_1=49.958$). Finally the author claims that by application of the criteria obtained by Merman (paper cited above) the occurrence of capture in the problem under consideration is to be considered as proved. *E. Leimanis* (Vancouver, B. C.).

Cavedon, Mario, ed Alessio, Virginia. Ricerche teoriche e numeriche per un caso particolare del problema dei 3 corpi. Mem. Soc. Astr. Ital. (N.S.) **25**, 421–430 (1 plate) (1954).

Two particles P_1 and P_2 which attract each other according to the Newtonian law are subject to the attraction of a third particle P_3 with a force proportional to the distance. Conditions are determined that the relative orbit of P_2 with respect to P_1 is a circle.

Choosing the units of time, length and mass and the initial conditions of motion to fit the situation in stellar systems, eight relative orbits of P_2 are calculated by the Runge-Kutta method of numerical integration. Prior to this the formal identity of differential equations for the following two relative motions is established: (i) in the above mentioned particular case of the three-body problem and (ii) for the motion of P_2 with respect to P_1 in the n-body problem in which these particles (attracting each other according to the Newtonian law) are subject to the attractions of $n-2$ particles $P_3, \cdots, P_n$ with forces varying proportionally to the distance [cf. also F. Zagar, Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15(84)**, 610–618 (1951); MR **15**, 260]. *E. Leimanis* (Vancouver, B. C.).

Neyman, Jerzy, and Scott, Elizabeth L. Spatial distribution of galaxies—analysis of the theory of fluctuations. Proc. Nat. Acad. Sci. U. S. A. **40**, 873–881 (1954).

Limber [Astrophys. J. **117**, 134–144 (1953); **119**, 655–681 (1954); MR **14**, 803; **15**, 994] has suggested a method for analyzing the counts of galaxies in which the hypothesis of a fluctuating density field is introduced as in the current theories of turbulence. The authors point out that in view of the discreteness of the substratum care is necessary and that the number of galaxies $n(S)$ in a region of space S is considered as a random variable which satisfies certain postulates of stationarity, additivity, etc. Also, that the correlation in the concentration of the galaxies at two points introduced in Limber's work should be redefined as follows:

For almost all positive ξ and for any arbitrary sequence of pairs of cubes $\{C(P, V_m), C(Q, W_m)\}$ with a distance $\xi>0$ between their centers P and Q and with volumes V_m and W_m, respectively, if

$$\lim_{m\to\infty} V_m = \lim_{m\to\infty} W_m = 0,$$

then there exists a finite limit $\Gamma(\xi)$ of the quotient of the correlation R_m between $n[C(P, V_m)]$ and $n[C(Q, W_m)]$ divided by the square root $\sqrt{(V_m W_m)}$,

$$\lim_{m\to\infty} \frac{R_m}{\sqrt{(V_m W_m)}} = \Gamma(\xi).$$

If $\sigma^2_{P_m}$ and $\sigma^2_{Q_m}$ stand for variances of $n[C(P, V_m)]$ and $n[C(Q, W_m)]$, respectively, then the covariance of these variables is $\sigma_{P_m}\sigma_{Q_m}R_m$. *S. Chandrasekhar.*

***Neyman, J., and Scott, E. L. On the problem of expansion of clusters of galaxies.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 336–345. Academic Press Inc., New York, 1954. $9.00.

The earlier considerations of the authors [Neyman and Scott, Astrophys. J. **116**, 144–163 (1952); MR **14**, 803; Neyman, Scott, and Shane, ibid. **117**, 92–133 (1953)] on the theory of the spatial clustering of the galaxies is generalized to allow for a sharing by the clusters in the general expansion of the universe. *S. Chandrasekhar* (Williams Bay, Wis.).

Sen, K. K. On the problem of softening of radiation by multiple Compton scattering in stellar atmospheres containing free electrons. Proc. Nat. Inst. Sci. India **20**, 530–541 (1954).

The problem of the title, set up as an integro-differential equation, was worked out in a first-order approximation by Chandrasekhar [Proc. Roy. Soc. London. Ser. A. **192**, 508–518 (1948); MR **9**, 512]. The present article extends it to the second-order approximation and makes use of trigonometric-series expansions instead of the Green's function as employed in the earlier paper. This second-order theory materially reduces the error occurring in a part of the intensity distribution curve arising in the first-order theory. *R. G. Langebartel* (Urbana, Ill.).

RELATIVITY

Szekeres, G. New formulation of the general theory of relativity. Phys. Rev. (2) **97**, 212–223 (1955).

The author points out a number of difficulties of interpretation in general relativity theory, concerning the principle of equivalence, the concept of cosmic time, and Mach's principle of the relativity of inertia. Generalizing an idea of Hoyle's [Monthly Not. Roy. Astr. Soc. **108**, 372–382; **109**, 365–371 (1949)], he proposes to resolve these difficulties by introducing into the field equations a scalar field τ which is to be identified physically as the cosmic time. The reference system defined by $C_m=\partial\tau/\partial x^m$ (Latin letters run over 0, 1, 2, 3) is called "the ether"; it specifies a state of "absolute

rest" in space-time. Field equations are derived from the variational principle

$$\delta\int(R^r{}_s+\tfrac{1}{2}\beta C^r{}_s C^s{}_r-\gamma\tau^{-2})\Delta^{\frac{1}{2}}d^4x=0,$$

where $R^r{}_s$ is the Ricci tensor, $\Delta=-\det(g_{mn})$, $C_{rs}=C_{sr}=C_{r;s}$, β is the coupling constant between gravitational and cosmic time fields, and γ is the cosmical constant (a semicolon denotes covariant differentiation). The field equations are $\beta C^{mn}{}_{;m}+2\gamma\tau^{-3}=0$ and $R_{mn}-\frac{1}{2}g_{mn}R^r{}_s+\beta T_{mn}+\frac{1}{2}\gamma\tau^{-2}g_{mn}=0$, where

$$(1)\quad T_{mn}=-\tfrac{1}{2}(C_m C_n{}^r{}_{;r}+C_n C_m{}^r{}_{;r})+\tfrac{1}{2}(C_{mn}C^r+C_{mn;r}C^r)-\tfrac{1}{4}g_{mn}C^r{}_s C^s{}_r.$$

These equations are treated from the point of view that there are no other sources of the gravitational field except the terms (1) arising from C_m and its derivatives. There are cosmological solutions $ds^2=dt^2-t^\alpha(dx^2+dy^2+dz^2)$, where α is a constant determining β and γ by the equations $\beta=8(3\alpha+2)^{-1}$, $\gamma=\frac{1}{4}\alpha^2(3\alpha-4)(3\alpha+2)^{-1}$. There are approximate solutions which are spherically symmetric and static, with $\gamma=0$. These lead to a planetary perihelion advance of $\{1-\beta(12+6\beta)^{-1}\}$ times the general relativity value. For $\beta=1$, a choice preferred by the author, this is in as good agreement with observation as is the general relativity value.

The author defines a vector density of gravitational energy-momentum, and using it, effectively rediscovers the general relativity result [E. T. Whittaker, Proc. Roy. Soc. London. Ser. A. **149**, 384–395 (1935); J. L. Synge, Proc. Edinburgh Math. Soc. (2) **5**, 93–102 (1937)] that under certain conditions $R_{rs}\lambda^r\lambda^s$ may be identified (aside from a constant factor) as the density of gravitational mass relative to a unit vector field λ^r. Choosing for λ^r the field $C^r(C_sC^s)^{-\frac{1}{2}}$, the author deduces certain results which he interprets to mean that his theory is irreconcilable with the principle of equivalence. A number of other consequences of the theory are discussed. *F. A. E. Pirani* (Dublin).

Papapetrou, A. Eine Theorie des Gravitationsfeldes mit einer Feldfunktion. Z. Physik **139**, 518–532 (1954).

In two previous papers [Math. Nachr. **12**, 129–141, 143–154 (1954); MR **16**, 634] the author sought a theory of gravitation based on a Riemannian metric of the form (1) $ds^2=e^\chi dt^2-e^\phi(dx^2+dy^2+dz^2)$, where χ and ϕ are functions of the coordinates. His object was to devise a theory embodying, in the form of a preferred class of coordinate systems, some generalization of the inertial reference frames of Newtonian mechanics, in order to circumvent difficulties of interpretation in general relativity theory concerning the energy-momentum pseudo-tensor of the gravitational field, the existence of gravitational waves, and other difficulties arising from the general covariance of that theory [cf., e.g., L. Infeld, Acta Phys. Polon. **13**, 187–204 (1954); MR **16**, 531; and the paper reviewed above].

The author now attempts to restrict further the transformation group under which his theory is covariant by considering only those metrics (1) which are of the form (2) $ds^2=e^{-\phi}dt^2-e^\phi(dx^2+dy^2+dz^2)$. From a variational principle like that of general relativity he derives the field equation $2R^0{}_0=\kappa(T^0{}_0-T^i{}_i)$ which he supplements with the dynamical equation $T_\mu{}^\nu{}_{;\nu}=0$ [Greek indices run 0–3; Latin indices 1–3]. He investigates infinitesimal coordinate transformations preserving the form of (2); he conjectures that unless $\phi=$constant, only 3-space rotations and certain changes of scale are admissible finite transformations. Among other consequences of the theory is the appropriate planetary perihelion motion prediction.

F. A. E. Pirani (Dublin).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. B_3. J. Rational Mech. Anal. **4**, 247–277 (1955).

Chronologically this is the latest of the author's main series of papers on the unified theory, the earlier ones being indexed by A, B, C, C_2, C_3, but its contents relate most closely to those of paper B [same J. **2**, 1–52 (1953); MR **14**, 505].

The connection $\Gamma^\nu{}_{\lambda\mu}$ of the unified theory is expressible in the form

$$\Gamma^\nu{}_{\lambda\mu}=\{{}^\nu_{\lambda\mu}\}+S_{\lambda\mu}{}^\nu+U^\nu{}_{\lambda\mu},$$

where the Christoffel symbols are formed from the symmetric part $h_{\lambda\mu}$ of the fundamental tensor $g_{\lambda\mu}$, $S_{\lambda\mu}{}^\nu$ is the torsion-tensor and

$$U^\nu{}_{\lambda\mu}=2h^{\nu\alpha}S_{\alpha(\lambda}{}^\beta k_{\mu)\beta},$$

$k_{\lambda\mu}$ being the skew-symmetric part of $g_{\lambda\mu}$. The tensor $S_{\alpha\beta\gamma}$ is in turn expressible in the form

$$S_{\alpha\beta\gamma}=\tfrac{1}{2}K_{\omega\rho\sigma}Y^{\omega\rho\sigma}_{\alpha\beta\gamma},$$

where $K_{\omega\rho\sigma}$ is a certain linear combination of covariant derivatives of $k_{\lambda\mu}$, and $Y^{\omega\rho\sigma}_{\alpha\beta\gamma}$ is a tensor of which the construction was described in paper B but of which the actual form in terms of $g_{\lambda\mu}$ was not given. The main object of the paper now under review is to obtain explicit forms of the tensors S, U, Y in terms of $g_{\lambda\mu}$. This is done by the use of non-holonomic frames having an intrinsic relationship with the tensors $h_{\lambda\mu}$, $k_{\lambda\mu}$, distinct cases arising according as $g_{\lambda\mu}$ belongs to one or other of three algebraic classes. *H. S. Ruse.*

Kilmister, C. W., and Stephenson, G. An axiomatic criticism of unified field theories. I, II. Nuovo Cimento 11, supplemento, 91–105, 118–140 (1954).

In the first part the authors consider a set of axioms for macroscopic unified theory. The first axiom rules out all p-dimensional geometries, $p>4$. Axiom II runs as follows: The space V of events consists of points; each point has a neighbourhood, the points of which are in 1–1 correspondence with the points of a Euclidean sphere. The third axiom restricts physically significant quantities to these expressed as tensors (densities). Axiom IV: In the absence of fields, the appropriate description of the theory is that of the special theory of relativity. According to the fifth axiom an appropriate description of a field whose first approximation is a gravitational field is that of general theory of relativity. Axiom VI restricts the generality by requiring that the function $f^i(x,\dot{x})$ in the description of motion $\ddot{x}^i=f^i(x,\dot{x})$ be a polynominal in $\dot{x}$, its coefficients being functions of x only. Of course according to Axiom VII in a true unified theory the unification of both fields (i.e. gravitational and electromagnetic) is not to be a purely formal one. Axiom VIII. The field variables are determined by a set of partial differential equations. These axioms according to the authors may be looked upon as conditions which a unified theory could reasonably be expected to satisfy. All such theories may be split in two classes which may be characterized roughly as follows: Theories of type (1) have a Riemannian underlying space, those of type (2) have a non-Riemannian space. The type (1) may be split in two classes, i.e. (11) and (12). In the first one the metric of the space describes the gravitational field, in the latter this is not the case.

In the second paper the authors deal with present unified theories from the point of view of the above mentioned axioms, such as affine theory of type (1), affine theory of type (2), non-affine theory and others. They reach the conclusion that none of these theories which satisfy the Axioms I–VIII are fully satisfactory. However the latest version of the Einstein unified theory seems to the authors to represent the most hopeful generalization of the general relativity. Many interesting details of this long paper must be omitted in this short review. *V. Hlavatý.*

Stephenson, G. Some properties of non-symmetric unified field theories. Nuovo Cimento (9) **12**, 279–284 (1954).

This paper contains three separate sections. In the first section the author expresses the symmetric part $\Lambda^{\nu}{}_{\lambda\mu}=\Gamma^{\nu}{}_{(\lambda\mu)}$ of the unified field connection $\Gamma^{\nu}{}_{\lambda\mu}$ by means of its skew symmetric part $S_{\lambda\mu}{}^{\nu}=\Gamma^{\nu}{}_{[\lambda\mu]}$ and vice versa. Then he identifies the electromagnetic field with $k_{\lambda\mu}=g_{[\lambda\mu]}$ and imposes on it the first set of Maxwell conditions

$$\partial_{[\nu}k_{\mu\lambda]}=0. \tag{1}$$

The second set of Maxwell conditions is equivalent to the Einstein condition $S_{\lambda\alpha}{}^{\alpha}=0$. However, according to the author, there appears to be no definite reason for imposing the additional condition (1). In the second part the author considers Einstein's condition

$$R_{[\mu\lambda]}=2\partial_{[\mu}X_{\lambda]} \tag{2}$$

coupled with

$$R_{[\mu\lambda]}=-\hat{D}_{\alpha}S_{\mu\lambda}{}^{\alpha}$$

(where $\hat{D}$ denotes the covariant derivative with respect to $\Lambda^{\nu}{}_{\lambda\mu}$). Hence

$$S_{\mu\lambda}{}^{\nu}=2X_{[\mu}\delta^{\nu}{}_{\lambda]}+T_{\mu\lambda}{}^{\nu}, \tag{3}$$

where $T_{\mu\lambda}{}^{\nu}=T_{[\mu\lambda]}{}^{\nu}$ is a solution of

$$D_{\alpha}T_{\mu\lambda}{}^{\alpha}=0. \tag{4}$$

Therefore $S_{\lambda\alpha}{}^{\alpha}=0$ is equivalent to

$$X_{\mu}=-\tfrac{1}{4}T_{\mu\alpha}{}^{\alpha} \tag{5}$$

and the field equations reduce to 16 equations (4) and $R_{(\mu\lambda)}=0$. In the third part the author considers all possibilities of defining $\Gamma^{\nu}{}_{\lambda\mu}$ by means of the derivatives of $g_{\lambda\mu}$ with all possible combinations of Einstein's signs $(++)$, $(+-)$, $(--)$. He concludes that in both cases (i.e. for real or complex $g_{\lambda\mu}$) only the $(+-)$ derivation leads to a connection $\Gamma^{\nu}{}_{\lambda\mu}$ without imposing severe restrictions on $g_{\lambda\mu}$. *V. Hlavatý* (Bloomington, Ind.).

Stephenson, G. Dirac's electrodynamics and Einstein's unified field theory. Nuovo Cimento (9) **10**, 1595–1596 (1953).

(See the above review.) The purpose of this note is to show that by imposing another condition on S_{λ} one obtains another electromagnetic theory. Instead of assuming $S_{\lambda}=0$, the author assumes

$$k_{\mu\lambda}=2\partial_{[\mu}S_{\lambda]} \tag{1}$$

$$S_{\lambda}S^{\lambda}=\text{const.} \tag{2}$$

The first equation is equivalent to the first Maxwell set, the second equation corresponds to Dirac's gauge-destroying condition $A_mA^m=\text{const.}$ [see P. A. M. Dirac, Proc. Roy. Soc. London. Ser. A. **209**, 291–296 (1951); MR **13**, 893]. From (2) one obtains for weak fields the second set of electromagnetic equations

$$S^{\nu}=\nabla_{\alpha}k^{\nu\alpha} \tag{3}$$

(where ∇ is the symbol of covariant derivative with respect to the Christoffel symbols of $h_{\lambda\mu}=g_{(\lambda\mu)}$). This equation is the generalization of the Dirac electrodynamic equation [Dirac, loc. cit.].

Remarks of the reviewer: I. The requirement (1) could be satisfied only if a certain set C of six differential equations of the second order (with respect to $k_{\lambda\mu}$) are satisfied. These conditions C have to be looked upon as additional field equations. II. The equations (3) for weak fields follow from the Einstein equations defining $\Gamma^{\nu}{}_{\lambda\mu}$ regardless of whether (2) holds or not. III. Whenever (1) holds, then $R_{\nu\mu\alpha}{}^{\alpha}=k_{\nu\mu}$. These remarks follow at once from (2.12) in Hlavatý, Nieuw Arch. Wisk. (3) **2**, 103–114 (1954) [MR **16**, 408] and in I (6.4)c in Hlavatý, J. Rational Mech. Anal. **3**, 103–146 (1954) [MR **15**, 654]. *V. Hlavatý.*

Winogradzki, Judith. Sur les équations du champ généralisé d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **240**, 945–947 (1955).

On considère les équations du champ qui résultent du principe variationnel le plus simple. Si l'on considère g_{ik} comme le champ généralisé, les équations auxquelles satisfait ce champ sont les conditions de compatibilité des équations déterminant $\Gamma^{i}{}_{km}$ et ont un caractère d'invariance, essentiel, que ne possèdent pas les équations d'Euler. Si l'on considère g_{ik}, Γ_{i} comme le champ généralisé, le formalisme est presque classique. *Author's summary.*

Lenoir, Marcel. Equations approximatives de la théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **240**, 1400–1402 (1955).

The author outlines an approximation procedure for the solution of the equations of the recent Einstein unified field theory [Supplement to Appendix II of "The meaning of relativity, 4th ed.", Princeton, 1953; MR **15**, 357] in terms of the symmetric tensor $l_{\alpha\beta}$ identified by Lichnerowicz [J. Rational Mech. Anal. **3**, 437–521 (1954); MR **16**, 408] as the gravitational field. *F. A. E. Pirani* (Dublin).

Pirani, F. A. E. On the energy-momentum tensor and the creation of matter in relativistic cosmology. Proc. Roy. Soc. London. Ser. A. **228**, 455–462 (1955).

It is assumed that the continual creation of material particles is accompanied by the simultaneous creation of entities of zero rest-mass and negative energy, called gravitinos, in such a way that 4-momentum is concerned at each event. In an expanding model universe it is asserted that the newly created matter "naturally" partakes of the motion of the already existing matter, though why this should be more "natural" than its opposite is not explained. The energy-momentum tensor for a mixture of pressure-less matter, together with gravitinos, is set up and introduced into Einstein's equations of gravitation. The contributions of the gravitinos is shown to give rise to McCrea's zero-point stress and also to a zero-point energy. The aim of the investigation is to save Einstein's equations from the kind of modification proposed by Hoyle, by altering the energy-momentum tensor rather than the geometrical tensor that occurs in them. It is speculated that the gravitinos could be regarded as neutrinos of negative energy. It is not evident to this reviewer why the introduction of the artificial concept of a gravitino should be an improvement in the theory of continual creation or be preferable to Hoyle's modification of Einstein's equations. *G. C. McVittie.*

Papapetrou, A., und Urich, W. Das Pol-Dipol-Teilchen im Gravitationsfeld und elektromagnetischen Feld. Z. Naturf. **10a**, 109–117 (1955).

This paper starts with Einstein's equations, with a particle tensor T_β^α differing from zero along a "world tube". In the final calculations the "world tube" shrinks to a world line. This tensor together with the tensor for the electromagnetic field is the energy-momentum tensor. The conditions put on the momenta of this tensor define the pole-dipole structure of the particle. The spin tensor is defined by means of the tensor T_β^α.

From the fact that the equations of motion are contained in the field equations, the covariant equations for the spin tensor and four-velocity of the particle are deduced. Making use of the canonical formalism, the possible forms of the spin tensor have been reduced to an expression with one arbitrary function only. The classical Poisson brackets of the theory are examined.

The whole throws light upon the problem of the pole-dipole particle in given gravitational and electromagnetic fields. Some doubt can be raised concerning the use of world tubes and the limiting processes, instead of functions.

J. Plebański (Warsaw).

Pham Mau Quân. Les équations du champ pour un schéma fluide-champ électromagnétique. C. R. Acad. Sci. Paris **240**, 598–600 (1955).

The following equations are assumed for an electromagnetic fluid:

$$\nabla_\alpha G^{\alpha\beta} = J^\beta, \quad \nabla_\alpha H^{*\alpha\beta} = 0, \quad J^\beta = \delta u^\beta + \Gamma^\beta,$$

$$u^\alpha G_{\alpha\beta} = l u^\alpha H_{\alpha\beta}, \quad m u^\alpha G^*_{\alpha\beta} = u^\alpha H^*_{\alpha\beta}, \quad \Gamma^\beta = \sigma u_\alpha H^{\alpha\beta}.$$

Here $G_{\alpha\beta}(=\mathbf{D}, \mathbf{H})$ and $H_{\alpha\beta}(=\mathbf{E}, \mathbf{B})$ are electromagnetic tensors, star indicates dual, J^β = 4-current, u^β = 4-velocity, δ = charge density, Γ^β = conduction current ($\Gamma^\beta u_\beta = 0$), l = dielectric constant, m = magnetic permeability, σ = electric conductivity. An unsymmetric electrodynamic tensor is defined by

$$\tau^{\alpha\beta} = \tfrac{1}{4} g^{\alpha\beta} G^{\lambda\mu} H_{\lambda\mu} - G^{\alpha\lambda} H_\lambda^\beta.$$

Generalising a previous formula [same C. R. **236**, 2299–2301 (1953); MR **14**, 1134], the author defines the energy tensor to be

$$T^{\alpha\beta} = \rho u^\alpha u^\beta - \pi^{\alpha\beta} - (u^\alpha q^\beta + u^\beta q^\alpha) + \tau^{\alpha\beta} - (1 - lm)\tau^{\alpha\rho} u_\rho u^\beta,$$

where ρ = proper density, $\pi^{\alpha\beta}$ = stress tensor ($\pi^{\alpha\beta} u_\beta = 0$), and q^α = heat-flow vector, satisfying

$$q^\alpha = -\kappa \partial_\rho \theta (g^{\rho\alpha} - u^\rho u^\alpha), \quad \nabla_\alpha q^\alpha = c^* \rho u^\alpha \partial_\alpha \theta - l^* u^\alpha \partial_\alpha \rho / \rho,$$

where θ = temperature, κ = thermal conductivity, c^* = specific heat at constant volume, l^* = heat of dilatation. The equations of motion are given by the conservation equations $\nabla_\alpha T^{\alpha\beta} = 0$. [The author does not state the important fact that, by virtue of the relations between u^α, $G_{\alpha\beta}$ and $H_{\alpha\beta}$, the tensor $T^{\alpha\beta}$ is symmetric.] *J. L. Synge* (Dublin).

Pham Mau Quân. Le problème de Cauchy relatif à un schéma fluide-champ électromagnétique. C. R. Acad. Sci. Paris **240**, 733–735 (1955).

The theory of the paper reviewed above is specialised to the case of a perfect fluid, for which $\pi^{\alpha\beta} = p(g^{\alpha\beta} - u^\alpha u^\beta)$ and there is an equation of state $\rho = \phi(p, \theta)$. Einstein's equations $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}$ are assumed. Using the method of Lichnerowicz [Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955], the author investigates the Cauchy problem, the values of $g_{\alpha\beta}$, $\partial_\lambda g_{\alpha\beta}$, θ, $\partial_\lambda \theta$, $H_{\alpha\beta}$ being assigned on a hypersurface S. He concludes that these data determine $g_{\alpha\beta}$, θ, $H_{\alpha\beta}$ in the neighbourhood of S, except for special choices of S. Details are to appear later, with a study of exceptional cases. *J. L. Synge.*

Hennequin, Françoise. Sur l'approximation des équations de la Relativité générale pour un champ quasi galiléen. C. R. Acad. Sci. Paris. **239**, 1464–1466 (1954).

This paper deals with the motion of a set of small bodies, each composed of perfect fluid; it appears that, to the order of approximation considered, it is not necessary to specify a connection between pressure and density. As basis of approximation (apart from the smallness of the bodies), the author expands $G^{\lambda\mu}$ [$= g^{\lambda\mu}(-g)^{1/2}$] in inverse powers of c (light velocity), and uses the coordinate conditions $\partial_\alpha G^{\alpha\lambda} = 0$. She obtains equations of motion to order c^{-2}, and states that these equations simplify in the case of spherical symmetry, but that they differ from those given by Papapetrou [Proc. Phys. Soc. Sect. A. **64**, 57–75 (1951); MR **12**, 546], who used a slightly different definition of mass. There appear to be a number of misprints and the argument is difficult to follow in this condensed presentation, in view of the delicate approximations involved. *J. L. Synge* (Dublin).

Balazs, Nandor L. The propagation of light rays in moving media. J. Opt. Soc. Amer. **45**, 63–64 (1955).

Let g_{ik} be the fundamental tensor of a gravitational field in a transparent medium which has 4-velocity u^i and index of refraction $n = (\epsilon\mu)^{1/2}$. For the purposes of the present paper, the histories of light rays in the medium are null geodesics for the tensor

$$\gamma_{ik} = g_{ik} + (1 - 1/n^2) u_i u_k.$$

For stationary cases the rays satisfy Fermat's principle; when there is no gravitational field and the velocity of the medium (v_α, $\alpha = 1, 2, 3$) is small (v^2/c^2 neglected), this principle may be written $\delta \int dl/W = 0$, where dl is the element of length and

$$W = c/n + (1 - 1/n^2) v_\alpha \lambda_\alpha, \quad \lambda_\alpha = dx_\alpha/dl.$$

J. L. Synge (Dublin).

Ueno, Yoshio. On the wave theory of light in general relativity. II. Light as the electromagnetic wave. Progr. Theoret. Phys. **12**, 461–480 (1954).

[For part I see same journal **10**, 442–450 (1953); MR **15**, 655.] The author discusses fields of electromagnetic radiation in a statical spherically symmetric (SSS) gravitational field with

$$ds^2 = -A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + C(r)dt^2,$$

the functions A and C being at first arbitrary and their Schwarzschild values being inserted later. He ignores the gravitational effects of the electromagnetic field. Maxwell's equations are used with the physical components E_r, E_θ, E_ϕ, H_r, H_θ, H_ϕ, and three types of waves are discussed: TE with $E_r = 0$, TM with $H_r = 0$, and TEM with $E_r = H_r = 0$. For TE and TM waves, the field depends on the solution of a scalar wave equation; for TEM waves it depends on a harmonic function of ϕ and $\log \tan \frac{1}{2}\theta$, and on arbitrary functions of $\sigma \pm t$, where $\sigma = \int (A/C)^{\frac{1}{2}} dr$. For waves of the form

$\psi=k(r,\theta,\phi)f(\sigma-t)$, where f is a periodic function, wave length, period, and phase velocity are defined, the last being in fact unity. Spectral shift and energy flow are discussed. Appendixes deal with the scalar wave equation in SSS, plane waves in SSS, physical components of vectors and tensors, a proof that any electromagnetic field in SSS is given by a suitable superposition of TE, TM, and TEM fields, and a discussion of relative dispersionless waves.

J. L. Synge (Dublin).

Glaser, Vladimir. New relativistic two-body equation. Phys. Rev. (2) **98**, 840 (1955).

MECHANICS

***Routh, Edward John. The advanced part of a treatise on the dynamics of a system of rigid bodies. Being part II of a treatise on the whole subject. 6th ed.** Dover Publications, Inc., New York, 1955. xiv+484 pp. Paperbound $1.95; clothbound $3.95.

Reprint by photo-offset of the 6th edition [Macmillan, London, 1905].

***Yourgrau, Wolfgang, and Mandelstam, Stanley. Variational principles in dynamics and quantum theory.** Sir Isaac Pitman & Sons, Ltd., London, 1955. ix+155 pp. $5.50.

This book contains an excellent account of variational principles in classical mechanics. The authors begin with a brief description of the ideas of Fermat and Maupertuis and then proceed to a discussion of the equations of mechanics as formulated by Lagrange and Hamilton. They also include an excellent summary of the Hamilton-Jacobi theory and its role in classical mechanics. The modifications produced in this theory by the advent of the special theory of relativity are treated very cursorily. An application of Hamiltonian methods to a continuous field is also given in the chapter on electrodynamics. However the exposition falters somewhat here since the authors do not make use of the notions of the theory of special relativity and in particular make no mention of the stress-energy tensor of the electromagnetic field. There are two chapters devoted to variational principles and quantum mechanics. The first of these gives an excellent summary of the Bohr-Sommerfeld atomic theory and the role that variational principles played in it. The second discusses the role that these principles played in the formulation of the Schrödinger form of quantum mechanics. However, the authors do not discuss the role that these principles have played in modern field theories and quantum mechanics. In fact, they state: "We for our part, contend that the function of action cannot be rated as a weighty factor in this field." The authors conclude with a chapter entitled "Significance of variational principles in natural philosophy." This chapter is somewhat critical of the approach of Planck to these principles in spite of the fact that the book is dedicated to him. *A. H. Taub.*

Block, H. D. On the minimality of the variational principles of classical particle mechanics. Amer. Math. Monthly **62**, 161–168 (1955).

A system of N particles in euclidean 3-space is subjected to certain forces: (1) forces derived from a potential function, (2) from constraints, and (3) other forces. It is shown that some of the classical variational principles are minimum principles under certain conditions and only stationary principles under other conditions. The author investigates the Hamilton principle, the principle of least action in the conservative case, Hamilton's modified principle and Hilbert's variational principle. It turns out that the two last-named principles cannot be stated as minimum (or maximum) principles. *J. Haantjes* (Leiden).

Grodzinski, Paul, and M'Ewen, Ewen. Link mechanisms in modern kinematics. Proc. Inst. Mech. Engrs. **168**, 877–888; discussion 889–896 (1954).

The paper contains twelve pages written by the authors, and eight pages of discussion involving nine participants. Only two of the topics will be mentioned here: (1) the seldom mentioned limitations on Grübler's criterion, and (2) a comparative table of the advantages and faults of linkages and cam mechanisms. The seven-line "Conclusion" states that "the study of linkages has for the past fifty years to date (1953) been practically a monopoly of German and Russian engineers". It is not clear how this conclusion was drawn since only three Russian titles are mentioned in the paper, one of them by Čebyšev (1853), and no mention is made of Dimentberg's work in spatial linkages. Moreover, to the best of this reviewer's knowledge, the recent Russian team work is done by people who call themselves applied mathematicians. Much of the discussion views with alarm the neglect suffered by the problem of classification and symbolization of mechanisms (so dear to Reuleaux). However, it is not explained why the problem is technologically important, nor is it mentioned that the formulation of this problem is a problem (if matrices are allowed for symbolization, and no metric information required, the solution is trivial). When are two mechanisms to be regarded as equivalent? In its simplest form (topological equivalence of electrical networks) the problem was raised already by Kirchhoff, and is still unsolved (except in the Pickwickian sense of the word).

A. W. Wundheiler (Chicago, Ill.).

Šul'gin, M. F. On Čaplygin's dynamical equations for nonintegrable constraining equations. Prikl. Mat. Meh. **18**, 749–752 (1954). (Russian)

The paper rederives the much-rediscovered Lagrangian equations for quasicoordinates and anholonomic constraints [Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed., Cambridge, 1937, p. 43] with unnecessary (Čaplygin) restrictions and in obsolete notation. [For a modern treatment of anholonomic coordinates see Schouten, Ricci-calculus, 2nd ed., Springer, Berlin, 1954, p. 100; MR **16**, 521.] The one-page derivation follows a three-page defense of Čaplygin's solution of the "sled" problem [a nonskidding thin wheel in contact with a plane; called a "knife-edge" in Hamel, Theoretische Mechanik, Springer, Berlin, 1949, p. 467; MR **11**, 548], using the distance traveled as a quasicoordinate, against the objections of M. I. Efimov [Prikl. Mat. Meh. **17**, 748–750 (1953); MR **15**, 659] that the equations apply to true generalized coordinates only. This defense takes the curious and confusing form of assertion that "redundant" coordinates may be used in the equations of motion. However, all the formulas involved can be interpreted if "quasicoordinates" are substituted for "redundant coordinates".

A. W. Wundheiler (Chicago, Ill.).

Bradistilov, G. The position of a system of three consecutively connected mathematical pendulums in one plane in their periodic motion about a position of stable equilibrium. Prikl. Mat. Meh. **19**, 113–118 (1955). (Russian)

The system consists of three heavy particles A_1, A_2, A_3, constrained to move in a vertical plane so that the distances OA_1, A_1A_2, A_2A_3, remain constant, O being fixed in the plane. For small oscillations about the vertical of O, the principal modes are determined, assuming $OA_1>A_1A_2>A_2A_3$. On the basis of a previously published theorem [Math. Ann. **116**, 181–203 (1938)], the author asserts that for every frequency of the linearized system which does not divide either of the two other ones, the exact equations will have a periodic solution with the same number of nodes although a different period. *A. W. Wundheiler* (Chicago, Ill.).

Bradistilov, G. The position of a system of three consecutively connected mathematical pendulums in one plane in their periodic motion about a position of stable equilibrium. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 135–145 (1954). (Bulgarian. Russian summary)

Translation of the paper reviewed above.

Rumyancev, V. V. Equations of motion of a rigid body having a cavity not completely filled with a fluid. Prikl. Mat. Meh. **18**, 719–728 (1954). (Russian)

Rumyancev, V. V. On the equations of motion of a rigid body with a cavity filled with a fluid. Prikl. Mat. Meh. **19**, 3–12 (1955). (Russian)

These papers are not very easy to interpret. Perhaps the reader will be served best if the following introductory remarks are made. If a mechanical system contains a rigid part R, and if O is any point, not necessarily solid with R, the six equations of momentum can be given the following Lagrangian form in quasicoordinates [cf. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed., Cambridge, 1937, p. 216]:

$$(24)\quad X_1=d(\partial T/\partial v_{10})/dt+\omega_2\partial T/\partial v_{30}-\omega_3\partial T/\partial v_{20},\ \cdots$$

$$(25)\quad L_1=d(\partial T/\partial\omega_1)/dt+\omega_2\partial T/\partial\omega_3-\omega_3\partial T/\partial\omega_2 +v_{20}\partial T/\partial v_{30}-v_{30}\partial T/\partial v_{20},\ \cdots,$$

where T is the entire kinetic energy of the system, v_{i0} the velocity of O, and ω_i the angular velocity of R. If the system consists of a rigid body and a perfect, incompressible fluid, the equations of motion of the latter may be conveniently written, relative to a R-bound frame, in the Euler form,

$$(26)\quad dv_1/dt+\omega_3v_2-\omega_2v_3=F_1-(\partial p/\partial x_1)/\rho,\ \cdots,$$

where F_i is the force per unit fluid mass. These equations constitute the result of the first paper. Why it was necessary to rederive them from the principle of minimum action, regarding the fluid as composed of a countable set of particles in certain equations, and as a continuum in others, is not clear to this reviewer.

In the second paper, the same equations are rewritten in terms of Lie's structure constants of the orthogonal group [following Poincaré, C. R. Acad. Sci. Paris **132**, 369–371 (1901); and Četaev, ibid. **185**, 1577–1578 (1927)], attaining a high degree of complexity (for instance, the numbers -1, 0, 1 are denoted by $c^{\nu}_{\alpha\beta\gamma}$, $\alpha,\beta,\gamma=1,\cdots,6$, $\nu=1,2,\cdots$, with varying positions of the upper indices), and regarding the fluid as a countable set of particles. The advantages of this form are not discussed. *A. W. Wundheiler.*

Grammel, Richard. Il giroscopio asimmetrico soggetto a momenti interni. Confer. Sem. Mat. Univ. Bari no. **2**, 20 pp. (1954).

A version of a paper previously published in German [Ing.-Arch. **22**, 73–97 (1954); MR **15**, 996]. *A. W. Wundheiler* (Chicago, Ill.).

Manacorda, Tristano. Una osservazione sulla dinamica dei fili. Boll. Un. Mat. Ital. (3) **9**, 385–390 (1954).

The following puzzling statement is proved: If a perfectly flexible and elastic cord moves in a plane so that all the velocities are parallel to a fixed direction x, then the initial conditions alone determine the motion, and even the load. Moreover, only one of the equations of motion is used in the proof. (Although it may be unfair to the reader, here is the well-hidden solution of the puzzle: there is no initial tension in the cord.) *A. W. Wundheiler* (Chicago, Ill.).

Charters, A. C. The linearized equations of motion underlying the dynamic stability of aircraft, spinning projectiles, and symmetrical missiles. NACA Tech. Note no. **3350**, 102 pp. (1955).

The purpose of this monograph is to obtain the equations of motion of aircraft and projectiles by a mathematical approach common to both cases. Aircraft are distinguished from projectiles and symmetrical missiles as follows: 1. Aircraft are assumed (a) to have a plane of mirror symmetry through the longitudinal axis and (b) to fly only slightly disturbed from a steady-state equilibrium attitude so that all components of the angular velocity of the aircraft are small. 2. Projectiles and symmetrical missiles are assumed (a) to have not only a plane of mirror symmetry but also 90° rotational symmetry (or its equivalent; i.e. rotational symmetry of order greater than 2) and (b) to fly similarly to aircraft except that the axial component of the angular velocity, the spin, may be large (with the restriction that the change in spin must be small). The two types of motion are treated similarly (but not simultaneously) with strictly similar mathematical developments and common nomenclature and coordinate systems. The resulting equations are not of course the same in the two cases because of the differences described above, nor are the types of motion similar in all respects. Thus the aircraft equations do not describe the gyroscopic nutation and precession of a spinning shell, and the projectile equations do not describe the phugoid oscillation of aircraft.

Because the departures from steady flight are assumed small it is possible to linearize the equations in the usual way by neglecting quantities of the second order of smallness. The aerodynamic coefficients are introduced as formal series expansions in the customary variables (effectively yaw, cross-spin, etc.) the coefficients depending only on the external shape and on fundamental aerodynamic parameters such as Mach and Reynolds numbers. The essential difference between the two cases lies in the fact that the projectile may have a large axial spin, so that for it terms have to be included which are negligible for an aircraft. On the other hand, the property of possessing rotational symmetry of order greater than 2 simplifies the projectile equations by providing relations between the various coefficients and so reducing considerably the number of parameters.

The equations obtained are solved and the stability of both types of motion considered briefly. They are also applied to the flight testing of rotationally symmetric projectiles in aerodynamic ranges. It is shown that the flight

records obtained in this way, while they do not provide estimations of all the parameters involved, can be analysed to give all the aerodynamic coefficients required to predict the first-order dynamic stability of the projectile.

The results are compared with the accepted aeronautical theory for aircraft [e.g., B. Melville Jones, Aerodynamic theory (W. F. Durand, ed.), Springer, Berlin, 1935, v. 5, pp. 1–222] and the corresponding ballistic theory for missiles and projectiles [e.g., E. J. McShane, J. L. Kelley and F. V. Reno, Exterior ballistics, Univ. of Denver Press, 1953; **15**, 657]. The agreement is considered to be satisfactory throughout, as the only discrepancies come from certain additional terms which are believed to be negligibly small under the majority of representative test conditions in the aerodynamic range.

The greater part of the mathematical analysis is carried out in the appendices where a careful examination is made of the order of magnitude of the various quantities neglected in the approximations and where the modifications necessary to include the effects of small aerodynamic asymmetries are also discussed. *R. A. Rankin* (Glasgow).

Hydrodynamics, Aerodynamics, Acoustics

Kravtchenko, Julien, and McNown, John S. Seiche in rectangular ports. Quart. Appl. Math. **13**, 19–26 (1955).

The authors' problem is to find a velocity potential of the form $\cosh k(z+h)F(x,y)$, where $(\nabla^2+k^2)F(x,y)=0$ in the rectangle $(0\leq x\leq a, 0\leq y\leq b)$; the normal gradient $\partial F/\partial n$ is to vanish at the boundary except on the segment $(\alpha\leq x\leq\beta, y=0)$, where $\partial F/\partial n=A$, a constant. This segment represents the entrance to the port, and it is stated that there is experimental evidence for the approximate validity of the boundary condition. It is shown that in principle a solution can be obtained in the form $F=f+u$ where f satisfies the boundary conditions but not the equation of continuity, while $\partial u/\partial n=0$ along the whole boundary. The function u can be expanded in an adequately convergent double series of eigenfunctions. When the forcing frequency coincides with a resonance of the closed rectangle this series requires modification. *F. Ursell* (Cambridge, England).

Zwick, S. A., and Plesset, M. S. On the dynamics of small vapor bubbles in liquids. J. Math. Phys. **33**, 308–330 (1955).

A very thorough theoretical study of heat transfer effects in spherical vapor bubbles in liquids. Using physical approximations justified in an earlier paper [J. Appl. Phys. **23**, 95–98 (1952); MR **13**, 751], the problem is reduced to a system consisting of a second-order differential equation and an integral equation. Using mathematical approximations, the radius of a slightly superheated bubble is shown to be nearly constant during an "early phase" of calculable length. This is fitted onto an "asymptotic solution" deduced in the paper cited above, by an analytical formula for the radius in the "intermediate phase", involving six parameters. These parameters are determined by solving six simultaneous non-linear equations. For collapsing bubbles, the system of equations is modified for physical reasons, and heat transfer effects shown to have very little effect on the radius-time curve. The mathematical techniques used are varied and ad hoc. *G. Birkhoff* (Cambridge, Mass.).

Isay, W.-H. Beitrag zur Potentialströmung durch radiale Schaufelgitter. Ing.-Arch. **22**, 203–210 (1954).

In a previous paper [Z. Angew. Math. Mech. **33**, 397–409 (1953)], Isay applied the theory of lifting surfaces to axial-flow blade cascades. In the present paper he applies the same theory to radial-flow blade cascades. He first considers the flow through infinitely thin blades rotating with an angular velocity ω. The absolute flow is a potential one, though the relative flow has a constant vorticity $-\omega$. This absolute flow is regarded as induced by a point source-vortex at the center of rotation and by a continuous distribution $\gamma(\varphi)$ of vortices along the blades, which are assumed to be represented by $r(\varphi)$, a polynomial or Fourier polynomial in φ. The requirement for the blades to be relative streamlines leads to an integral equation for $\gamma(\varphi)$. The results are then extended to cascades of thick blades as well as to a two stage system composed of a cascade of fixed (guide) vanes and a concentric cascade of rotating blades. *M. Marden* (Milwaukee, Wis.).

Fil'čakov, P. F. An engineering method of computing the filtration under the apron of hydraulic structures. Ukrain. Mat. Ž. **6**, 233–244 (1954). (Russian)

Chandrasekhar, S. The character of the equilibrium of an incompressible fluid sphere of variable density and viscosity subject to radial acceleration. Quart. J. Mech. Appl. Math. **8**, 1–21 (1955).

This paper is devoted to a consideration of the following problem: An incompressible fluid sphere, in which the density and the viscosity are functions of the distance r from the center only, is subject to a radial acceleration $-\gamma r$, where γ is a function of r; to determine the manner of initial development of an infinitesimal disturbance. By analysing the disturbance in spherical harmonics, the mathematical problem is reduced to one in characteristic values in a fourth-order differential equation, and a variational principle characterizing the solution is enunciated. The particular case of a sphere of radius R and density ρ_1 embedded in a medium of a different density ρ_2 (but of the same kinematic viscosity ν) is considered in some detail; and it is shown that the character of the equilibrium depends on the sign of $\gamma_R(\rho_2-\rho_1)$ and the magnitude of $G=\gamma_R R^4/\nu^2$. If $\gamma_R(\rho_2-\rho_1)>0$, the results of both an exact calculation and an approximate calculation (based on the variational principle) are given and contrasted. In the case $\gamma_R(\rho_2-\rho_1)<0$ when the situation is stable, the manner of decay of the disturbance is briefly discussed in terms of an approximate theory only. (Taken from the author's summary.)
R. C. DiPrima (Cambridge, Mass.).

Teng Fan, T. Y. The character of the instability of an incompressible fluid of constant kinematic viscosity and exponentially varying density. Astrophys. J. **121**, 508–520 (1955).

The equations determining the rate of growth of a horizontal periodic disturbance (of a given wave number k) in an incompressible fluid of constant kinematic viscosity ν and exponentially varying density ($\rho=\rho_0\exp[\beta z]$) are obtained. It is shown that this rate of growth depends on the two dimensionless parameters $L=l\beta$ and $G=g/\nu^2\beta^3$, where l is the depth of the fluid layer considered and g is the value of gravity. For a number of assigned values of L and G, the dependence of the growth rate on $\kappa=k/\beta$ is established, and the mode of maximum instability $\kappa=\kappa_{max}$ is determined.

For large or small values of G, simple asymptotic formulas for κ_{max} are derived. (Taken from the author's abstract.)

R. C. DiPrima (Cambridge, Mass.).

Hämmerlin, Günther. Uber das Eigenwertproblem der dreidimensionalen Instabilität laminarer Grenzschichten an konkaven Wänden. J. Rational Mech. Anal. **4**, 279–321 (1955).

In this paper the equations derived by Görtler [Nach. Ges. Wiss. Göttingen. Fachgruppe I. (N.F.) **2**, 1–26 (1940); MR **2**, 267] governing the stability of three-dimensional disturbances of a viscous incompressible flow over curved surfaces are treated by more refined techniques than previously. The problem involves the determination of the critical parameter $\mu=2(v_0\delta/\nu)^2\delta/R$ as a function of $\sigma=\alpha\delta$ for given values of $\tau=(\alpha^2\delta^2+\beta\delta^2/\nu)^{1/2}$. Here R is the radius of curvature of the wall taken constant and positive on the side concave to the flow and assumed great compared to the boundary-layer thickness δ, $\alpha=2\pi/\lambda$ where λ is the wave length of the disturbance parallel to the wall and perpendicular to the direction of mean flow, β is the rate-of-growth factor of the disturbance, and U_0 is the free-stream velocity. The case of neutral stability is given by $\beta=0$.

The author is able to show by an appropriate change of variables in the integral equations formulated by Görtler that there does exist for given β a least value of μ as a function of $\alpha\delta$, and that μ is positive. He then describes an iteration procedure for determining the least eigenvalue and applies it in the case $\beta=0$, i.e. the case of neutral stability. It appears that the minimum value of μ is taken on at $\alpha\delta=0$, and this is substantiated by solving the original system of differential equations when $\alpha\delta=0$. This result is in disagreement with Görtler's result that the minimum value of μ occurs for a finite value of $\alpha\delta$ obtained by an approximate solution of the integral equations. This result is obtained for several different mean-velocity profiles. The problem for $\beta\neq 0$ and a straight-line profile is treated directly by the method of power series from the differential equations. With the exceptions indicated above the results are in general agreement with those of Görtler; and hence it would appear that the results of Meksyn [Proc. Roy. Soc. London. Ser. A. **203**, 253–265 (1950); MR **12**, 451], who obtained a neutral curve higher by a factor of three than Görtler, using asymptotic solutions of the differential equations, are in error. *R. C. DiPrima* (Cambridge, Mass.).

Mager, Artur. Incompressible non-meridional boundary layer flow on bodies of revolution. U. S. Naval Ordnance Test Station, Inyokern, Calif., Rep. **NOTS 942**, v+53 pp. (1954).

The flow is supposed to involve a small perturbation of meridional flow over a rotating or non-rotating body of revolution. The laminar case is considered first. The boundary-layer flow in the meridional direction is unaffected by the other component, in the first approximation. The meridional potential-flow velocity component is here assumed to vary as a power of the distance measured along a meridian. The perturbation components are sought in power-series form. Two examples are treated: the flow over a cylindrical shell performing a turning (yawing) and spinning motion, and the flow against a revolving conical dish.

Turning to the turbulent boundary layer, the author employs the momentum-integral method. He assumes a simple wall-shear formula independent of pressure gradient and of three-dimensional effects; this shearing stress is taken in the direction of the flow at the body surface. He also assumes a one-parameter family of boundary-layer velocity profiles for both component flows. The unperturbed profile is a simple, fixed power law, so that the variable parameters for the meridional and azimuthal boundary-layer profiles are measures of the three-dimensional effects. They are determined by the two momentum-integral equations relating first-order terms, and thus by a pair of first-order differential equations. No solutions are worked out. The paper closes with a section showing how the "displacement surface" (analogous to the displacement thickness of plane flow) defined by F. K. Moore [NACA Rep. no. **1124** (1953)] can be determined, and a discussion of the applicability of the methods of this paper. *W. R. Sears.*

Cohen, Clarence B., and Reshotko, Eli. Similar solutions for the compressible laminar boundary layer with heat transfer and pressure gradient. NACA Tech. Note no. **3325**, 67 pp. (1955).

By requiring similarity and using Stewartson's transformation [Proc. Roy. Soc. London. Ser. A. **200**, 84–100 (1949); MR **11**, 553] the laminar compressible boundary layer equations are reduced to a set of ordinary nonlinear differential equations. Further restrictions that are made are that the Prandtl number be 1, the viscosity-temperature relationship be linear, the surface be isothermal, and the free-stream velocity be consistent with similar solutions. Under such restrictions it is possible to admit axial pressure gradients of arbitrary magnitude, heat flux normal to the surface, and arbitrary Mach numbers. The system of differential equations is transformed to a system of integral equations with the velocity ratio as independent variable. Solutions of this system are found numerically for a wide range of pressure gradients, from that causing separation to the infinitely favorable one, and wall temperatures, from absolute zero to twice the free-stream stagnation temperature. All solutions are presented in tabular and graphic form, and a discussion of the variance of such quantities as boundary layer thickness, skin friction, and heat transfer with changing pressure gradients and wall temperature is included. Also the uniqueness of the solution is discussed.

R. C. DiPrima (Cambridge, Mass.).

Low, George M. The compressible laminar boundary layer with fluid injection. NACA Tech. Note no. **3404**, 29 pp. (1955).

In this report, the equations of the compressible laminar boundary layer over a flat plate including the effects of transpiration cooling are studied. It is assumed that the normal velocity at the wall is proportional to $x^{-1/2}$ (here x is distance from the leading edge); that the wall temperature is constant (necessary for similar solutions); that the viscosity and temperature are linearly related; and finally that the Prandtl number and specific heat are constant. A similarity variable [Howarth, Proc. Roy. Soc. London. Ser. A. **194**, 16–42 (1949); MR **10**, 270] is introduced, and the boundary layer equations are reduced to a system of nonlinear ordinary differential equations. These equations are solved numerically for several different values of $f(0)$ where $f(0)$ is proportional to the mass flow of the fluid injected through the porous plate.

The effect of several flow parameters on coolant-flow rates is discussed with the aid of representative examples. A stability analysis indicates that, although transpiration cooling requires a lower surface temperature for stable flow

than does internal wall cooling, this lower temperature can be obtained with a smaller expenditure of coolant.

R. C. DiPrima (Cambridge, Mass.).

Rouleau, W. T., and Osterle, J. F. The application of finite difference methods to boundary-layer type flows. J. Aero. Sci. **22**, 249–254 (1955).

Stewartson, K. On the motion of a flat plate at high speed in a viscous compressible fluid. II. Steady motion. J. Aero. Sci. **22**, 303–309 (1955).

This is a continuation of the author's previous paper on the same subject [same Proc. **51**, 202–219 (1955); MR **16**, 638] extended to steady motion. In the region $M^3\nu_0/U<x<M^6\nu_0/U$, where x is the distance from the leading edge and M, ν_0 and U are respectively the Mach number, kinematic viscosity and velocity of the free stream; similarity solutions, satisfying the no-slip condition at the wall and conditions at the shock, are obtained for both inviscid and viscous regions. Finally, in the weak interaction region, the effect of the entropy in inviscid region is examined. *Y. H. Kuo* (Ithaca, N. Y.).

Ames Research Staff. Equations, tables, and charts for compressible flow. NACA Rep. no. **1135** (1953), iii+69 pp. (1954).

Ce rapport, d'une utilité indiscutable, contient d'abord une liste des équations les plus employées dans l'étude des fluides compressibles aussi bien dans le cas d'un écoulement continu que dans le cas où des ondes de choc sont présentes. Le cas où le fluide n'obéit pas aux lois des gaz parfaits est envisagé. Des tables très détaillées fournissent les valeurs des rapports utilisés dans les applications lorsque le fluide est parfait (écoulements continus, choc normal). Les résultats numériques relatifs aux chocs obliques sont traduits par des planches à grande échelle permettant d'obtenir trois chiffres significatifs. Enfin d'autres planches permettent d'obtenir les valeurs numériques des grandeurs les plus intéressantes dans les applications lorsqu'on ne peut plus admettre que les chaleurs spécifiques restent constantes.

P. Germain (Paris).

Woods, L. C. Subsonic plane flow in an annulus or a channel with spacewise periodic boundary conditions. Proc. Roy. Soc. London. Ser. A. **229**, 63–85 (1955).

Ce mémoire contient d'abord l'énoncé de deux théorèmes donnant les valeurs d'une fonction analytique définie à l'intérieur d'un rectangle connaissant certaines données sur les cotés du rectangle. Les résultats sont exprimés sous une forme adaptée aux applications; celles ci sont relatives essentiellement à certains écoulements d'un fluide incompressible (les résultats pouvant être étendus au cas d'un fluide compressible si l'on admet l'approximation classique de Chaplygin-von Karman). La méthode consiste à déterminer d'abord le potentiel complexe de l'écoulement dans le plan de l'hodographe; cette détermination est assez aisée lorsque le long des lignes bordant la région occupée par le fluide, le vecteur vitesse garde soit une intensité constante (ligne d'égale pression) soit une orientation constante (paroi rectiligne). Si l'obstacle donné est limité par une paroi curviligne, on doit au préalable résoudre par approximations successives une équation intégrale. Le passage au plan physique se fait suivant la méthode classique. Parmi les exemples signalés, deux sont traités en détail, le premier considère l'écoulement dans une région annulaire (notamment entre deux rectangles), le second est relativ à l'écoulement d'un fluide à l'intérieur d'un rectangle, ce fluide entourant une bulle à pression constante. *P. Germain.*

Skripkin, V. A. On transsonic flow of a plane gas jet from a nozzle with parallel walls. Prikl. Mat. Meh. **19**, 89–98 (1955). (Russian)

The author considers oscillatory motion of a slightly supersonic jet into an atmosphere of slightly lower pressure. Entropy changes are ignored and equations of motion simplified by use of transonic approximation. Analytical solutions are developed in parts of the expanding half of the first wave of the jet. Solutions in the simple wave regions centered on the edges of the nozzle unit can be written down immediately and determine conditions on the upstream sides of the quadrilateral in which these centered waves interact. The motion in the region of interaction is then found in the hodograph plane by solving the Euler-Poisson equation by Riemann's method. In the general case the analytical solution is terminated at the downstream vertex of the quadrilateral, but, in the special case of a sonic jet, the analytical solution is extended to cover the whole of the expanding half of the wave.

The author incorrectly states that the flow pattern in the second, compressive half of the first wave is a mirror image of that in the expanding half. In fact, as is well known, this compression is achieved through shocks passing through nodal points in the free streamlines. Only the part of this motion ahead of the shock can be determined by reflection of the expanding flow pattern. *M. Holt* (Sevenoaks).

Cole, J. D. Acceleration of slender bodies of revolution through sonic velocity. J. Appl. Phys. **26**, 322–327 (1955).

Dans ce travail, l'auteur se propose de discuter la validité de la théorie linéaire en évaluant l'importance des termes non linéaires que l'on a négligés. La méthode consiste à former l'équation du potentiel des vitesses en conservant les termes d'ordre 2, à résoudre le problème dans le cadre de la théorie linéaire classique et à utiliser cette solution pour évaluer un nombre λ qui caractérisera la valeur maximum du rapport des termes non linéaires et des termes linéaires significatifs. Le calcul de λ est d'abord effectué dans le cas des mouvements permanents. Pour les écoulements autour de profils λ ne dépend que du paramètre de similitude transsonique. Dans la dernière partie de l'article, le cas d'un corps fuselé passant la vitesse du son avec une accélération uniforme est envisagé; le paramètre λ dépend alors de l'épaisseur relative du corps fuselé et d'un paramètre sans dimension qui est le rapport au carré de la celérité du son du produit de l'accéleration par la longueur de l'engin. Son expression complète est calculée lorsque la méridienne du corps fuselé est un arc de parabole. *P. Germain.*

Vincenti, Walter G., and Wagoner, Cleo B. Theoretical study of the transonic lift of a double-wedge profile with detached bow wave. NACA Rep. no. **1180** (1954), ii+24 pp. (1955).

Dans ce travail on étudie la portance d'un profil losangique dans le domaine transsonique lorsque l'onde de choc est détachée. L'épaisseur et l'incidence sont supposées faibles, afin qu'on puisse appliquer les équations simplifiées des écoulements transsoniques, qui conduisent dans le plan de l'hodographe à une équation de Tricomi à laquelle doit satisfaire la fonction de courant. Les conditions aux limites

sont complexes: le problème à résoudre est un problème de Tricomi généralisé, en ce sens que sur un arc de la frontière située dans la région subsonique (celui qui correspond à la polaire de choc de l'écoulement amont) on ne connait pas la fonction de courant mais une relation entre ses dérivées premières. La solution est calculée par la méthode de "relaxation" qui est utilisée dans la région où l'équation est du type elliptique après avoir remplacé les données connues dans la région supersonique par une relation intégrale le long de la ligne sonique. Nous ne pouvons ici rendre compte complètement de l'ingéniosité qu'il a fallu aux auteurs pour surmonter les difficultés inhérentes au problème afin de parvenir, après des simplifications et approximations légitimes à un calcul qu'il soit possible d'effectuer.

Les résultats montrent qu'au voisinage du nombre de Mach $M=1$, le coefficient de portance varie peu; par contre il présente un maximum assez accentué dans la région voisine du nombre de Mach auquel l'onde de choc s'attache au bord d'attaque. Pour confirmer ce résultat les auteurs ont calculé la portance dans le cas où l'écoulement est partout supersonique. *P. Germain* (Paris).

Ward, G. N. The drag of source distributions in linearized supersonic flow. Coll. Aeronaut. Cranfield. Rep. no. **88**, 14 pp. (1955).

Starting with a general expression for the drag due to supersonic sources within a volume, the author considers first thin (wing-like) and then slender bodies. In both cases the boundary condition at the surface can be satisfied by surface distributions of sources of strength proportional to the surface slope. The formula obtained for the drag of thin wings resembles Lighthill's result for a special class of plane wings [J. Aero. Sci. **16**, 69–83 (1949); MR **10**, 338]. The formula for the drag of slender bodies is a generalization of Lighthill's result for bodies of revolution with discontinuities of meridian section [Quart. J. Mech. Appl. Math. **1**, 90–102 (1948); MR **10**, 76].

The minimum drag of a slender fuselage with attached, prescribed, thin wings is considered next. It is shown that the problem can be reduced to that of minimizing the drag of a certain fictitious slender body whose area distribution (along the stream direction) is the sum of the real area distribution sought $S(z)$ and a distribution $A(z)$ of the given wing volume according to a prescription given here. The prescription is simply that every volume element of the wing is distributed (nonuniformly) over the part of the longitudinal body (z) axis intercepted between its forward and rearward Mach cones. Thus the wing volume is distributed over that part of the axis intercepted between the Mach cones from the wing tips. It appears, therefore, that the minimum drag of the wing-body combination at no lift is achieved when $S(z)+A(z)$ is distributed in the shape of a minimum-drag slender body, such as has been worked out by several authors. The total drag is then not equal to the drag of this fictitious slender body but can be computed in a straightforward manner. *W. R. Sears* (Ithaca, N. Y.).

Freeman, N. C. A theory of the stability of plane shock waves. Proc. Roy. Soc. London. Ser. A. **228**, 341–362 (1955).

Un piston constitué par un cylindre dont la surface est voisine d'un plan est animé subitement d'une translation rectiligne uniforme. L'auteur étudie l'onde de choc qui prend naissance; c'est un problème à deux dimensions. Les équations sont linéarisées. Dans le cas où le piston est un dièdre dont l'angle est voisin de π la solution s'exprime à l'aide des fonctions théta. Dans le cas d'un piston limité par le cylindre $x=\epsilon\cos\omega y$, les perturbations dûes à la forme du piston décroissent comme $t^{-3/2}$ lorsque le temps t augmente indéfiniment; l'onde de choc plane est donc stable, par rapport à ces perturbations. L'auteur envisage le cas des chocs violents et enfin le cas des chocs faibles pour lesquels la solution est obtenue au moyen des fonctions de Bessel. *H. Cabannes* (Marseille).

Sakurai, Akira. The flow due to impulsive motion of a wedge and its similarity to the diffraction of shock waves. J. Phys. Soc. Japan **10**, 221–228 (1955).

L'auteur remarque l'analogie du problème enoncé avec celui de la rencontre d'une onde de choc plane et d'un dièdre. Loin du sommet du dièdre l'écoulement et le choc sont déterminés par la théorie des mouvements rectilignes; au voisinage du sommet il convient de faire des approximations. L'auteur étudie d'abord le cas des petits angles: le choc est attaché ou détaché suivant la vitesse du dièdre. Les indications qualitatives sont données pour le cas des dièdres dont l'angle n'est pas petit. Ce travail doit être rapproché de celui analysé ci-dessus, ainsi que d'une étude analogue antérieure [H. Cabannes, Rech. Aéro. no. **36**, 7–12 (1953); MR **15**, 479]. *H. Cabannes* (Marseille).

Kawamura, Tôru. Contribution of the change of entropy to the directions of spines of shock polar. J. Phys. Soc. Japan **9**, 396–400 (1954).

Expressions are derived for the initial slopes in the hodograph plane of streamlines of a uniform stream which passes through a curved two-dimensional shock. The slopes are found as a function of the free-stream Mach number and the angle of inclination of the shock, first taking account of the entropy variation behind the shock and then neglecting this entropy change. Results are plotted as a function of the angle of inclination for the case of free-stream Mach number equal to 2 and a considerable deviation is found in the range in which the initial slope of the streamlines in the hodograph plane is 90°. The author concludes that in order to study the flow characteristics of the subsonic flow region behind a curved shock it is necessary to take proper account of the entropy variation. *P. Chiarulli* (Washington, D. C.).

D'yakov, S. P. Interaction of shock waves with tangential and weak discontinuities. Dokl. Akad. Nauk SSSR (N.S.) **99**, 921–923 (1954). (Russian)

The author classifies all important cases of interaction of shock waves with tangential discontinuities (i.e. vortex sheets and free streamlines), of one tangential discontinuity with another, and of weak waves with shock waves and tangential discontinuities respectively. Only regions of plane, steady, uniform flow are considered. *M. Holt.*

D'yakov, S. P. Shock waves in a relaxing medium. Ž. Eksper. Teoret. Fiz. **27**, 728–734 (1954). (Russian)

L'auteur étudie les conditions d'existence des ondes de chocs dans un milieu de relaxation. En se basant sur les recherches de J. B. Zeldovitch [même Ž. **16**, 365–368 (1946)], l'auteur expose une théorie analytique d'une processus de relaxation suivi d'une baisse de pression, en supposant que les ondes de choc sont de faible intensité. *M. Kiveliovitch* (Paris).

Kuessner, H. G. The difference property of the kernel of the unsteady lifting surface theory. J. Aero. Sci. **22**, 227–230 (1955).

Yur'ev, I. M. On the calculation of nozzles. Prikl. Mat. Meh. **19**, 103–105 (1955). (Russian)

The author solves the equations of potential flow in the neighbourhood of the throat of a plane channel or axially symmetrical nozzle by means of a transonic approximation. Using coordinates based on the centre of the throat section, (x, y) in plane flow, and (x, r) in axially symmetrical flow, the reduced potential function is assumed to be a quadratic function of x, the coordinate measured downstream along the axis of symmetry. The coefficient of x^2 is constant but other coefficients depending on y (or r) are determined from the equations of motion. Simple expressions for the velocity distribution and shape of linear are derived. *M. Holt.*

***Rudinger, George. Wave diagrams for nonsteady flow in ducts.** D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. xi+278 pp. $6.00.

This is a timely book which fills an acutely felt need for a simple treatise on non-steady motions of compressible fluids in ducts. The only criticism which one might advance, if it can be called criticism, is that the booklet is too short and not comprehensive enough. As stated in the Preface, the author's aim is "to prepare a consistent set of computing procedures to cover most of the problems of non-steady flow that ordinarily may be encountered" and "The text is so arranged that it can be used either as a convenient manual of required procedures or for study by a newcomer . . .". In the reviewer's opinion, this is much more modest an aim than one would like to see accomplished.

As stated in the Introduction, non-steady flow phenomena occur in many engineering devices, and are important in the following: non-steady-flow jet engines (pulsejets, intermittent ramjets); non-steady-flow gas turbines and gas generators ("Comprex"); exhaust systems for internal combustion engines (particularly dynamic scavenge systems in two-stroke engines); internal ballistics of guns; pulse starting of supersonic wind tunnels; stabilization of shock waves in supersonic diffusers; non-steady-flow in gas lines; wave phenomena resulting from combustion processes; shock tubes, as used for creation of supersonic and hypersonic flows, fundamental gas dynamics studies, study of shock waves, study of the kinetics of chemical reactions; and several others (e.g. electric circuit breakers, pneumatic brake systems) in addition. In fact, it should be realized that a steady state in a flow-type engineering device is reached after a transient period of non-steady flow started either from rest or from another steady state, and consequently the study of starting or regulating processes always involves the consideration of non-steady motions. It has been customary in the past to obtain solutions for such cases on the assumption that the flow is quasi-steady at each instant of the transition period, but such assumptions can lead to completely erroneous results unless the change is extremely slow.

The subject under consideration has undergone a period of rapid development during the last fifteen years, but the results of these investigations are widely scattered and often inaccessible. So far, there are in existence the well-known books by Courant and Friedrichs [Supersonic flow and shock waves, Interscience, New York, 1948; MR **10**, 637], and by Sauer [Ecoulements des fluides compressibles, Beranger, Paris-Liège, 1951; Aufangswertprobleme bei partiellen Differentialgleichungen, Springer, Berlin, 1952; MR **13**, 295; **14**, 559], which concentrate on the mathematical aspects of the subject. In general, engineers and research workers for whom the results of such analyses are important would find it difficult to master the subject by first studying the mathematical theorems about, and the mathematical techniques for solving the hyperbolic partial differential equations involved. Thus, the need for a treatise which would facilitate this, and which would enable the reader to apply the theory and the techniques to important engineering problems has been felt acutely for some time. In particular, a serious student would also appreciate clear statements of the type of problems which have, as yet, defied solution.

A general method of solving the simplest case of non-steady flow, involving one space co-ordinate and time, is comparatively simple to understand and to apply, but the aspect which usually baffles the beginner is a realistic formulation of the boundary conditions, such as e.g. the conditions at an open end of a tube, and the occurrence of travelling or stationary shock waves. The author has given a satisfactory discussion of different boundary conditions which may be encountered in practice as well as a thorough exposition of the graphical-analytical method of solution based on the position, or (x, t) diagram. It is, however, regrettable that so little use has been made of the state diagram, at least for illustrative purposes, as this, in the reviewer's experience, facilitates understanding. For this and other reasons it is doubtful whether a complete newcomer would be able to make a systematic study of the subject with the present textbook alone. He will, however, be in a position, once he has grasped the fundamentals, to organize his work efficiently. In this connection Ch. VIII on "Techniques of wave diagram construction", and Ch. IX containing "Examples", and the "Charts and tables" in Ch. XI are particularly well thought out and presented.

In deriving the fundamentals the author leaves many questions which would naturally present themselves to a beginner, unanswered. For example, the fundamental equation for the speed of propagation of small disturbances (speed of sound) is only stated (eq. (III c4) on p. 14) and not derived. There is no rigorous and consistent development of the theory with a gradual mounting of difficulties to be overcome. The author was probably aware of this, when he stated that "throughout the text, emphasis is placed on the practical aspects of carrying out the computations . . .".

The author is well acquainted with the outstanding problems which require attention and he judiciously points them out in the text. In particular, the following statement should be borne in mind by research workers now active in this field, as well as by engineers who wish to make practical applications: "As is typical for a field in a phase of rapid growth, derivations must sometimes be based on assumptions for which experimental verification is not yet available, and the tentative nature of the resulting procedures is stressed." In view of this, it is regrettable that no systematic comparison between experiment and theoretical results has been made.

The book is extremely well laid out and printed. It contains an adequate bibliography which, however, contains several significant omissions, such as e.g. references to the experimental work performed by G. F. Mucklow and others at Birmingham University in England [e.g. Bannister and Mucklow, Inst. Mech. Engrs. Proc. **159**, 269–300 (1948)], or the report by R. C. Roberts [Tech. Rep. no. **F-TR-1173D-ND (GDAM A-9-M II/2)**, Wright Field, Dayton, Ohio, 1947; MR **10**, 161]. The book seems to be free from misprints and only very small flaws have been

detected, such as the mention of the First Law of thermodynamics on p. 15 where, clearly, the Second Law is meant. In conclusion, it will form a valuable addition to the library of any research worker or engineer interested in non-steady compressible fluid flows. *J. Kestin* (Providence, R. I.).

Gordov, A. N. Temperature of an infinite cylinder in a flow with pulsating velocity and temperature. Prikl. Mat. Meh. **19**, 240–243 (1955). (Russian)

The paper deals with the problem of response of a thin cylindrical probe placed in a pulsating stream. The difference between the present and standard solution lies in the fact that the coefficient of heat transfer, α, has not been assumed constant. Postulating that the pulsating temperature $t(\tau)=t_m+A_1\cos\omega\tau$ (τ=time, t_m=mean temperature) is accompanied by a synchronously pulsating velocity, the author assumes that the Biot (i.e. Nusselt) number $hR=(\alpha/\lambda)R$ varies in the same manner:

$$hR=h_0R+A_2R\cos\omega\tau$$

(λ-thermal conductivity of fluid, R-outer radius of cylinder). No discussion of the physical basis of this, otherwise plausible, assumption is given.

The author then proceeds to solve the heat-conduction equation in cylindrical, polar co-ordinates (assuming, again without discussion of the physical implications, that $\partial/\partial\varphi=0$) for an infinite cylinder, where the body temperature $\theta(r,\tau)$. Having set up the appropriate initial and boundary conditions, the author solves the problem by the use of the Laplace transform method. The final solution (rather complicated) is obtained in the form of an infinite series, whose first five terms have been given explicitly for the solution for the mean temperature $\bar{\theta}(\tau)$ (the mode of averaging over the radius r is not indicated).

The following conclusions are drawn. 1. The mean cylinder temperature is shifted with respect to the mean fluid temperature t_m by a constant $M_1=\frac{1}{2}A_1A_2R/(h_0R+A_2R\,\mathrm{ber}\,HR)$, $H=(\omega/a)^{\frac{1}{2}}$, which is a measure of the inherent error in response. The influence of a phase angle φ_0 between the coefficient of heat transfer and temperature is also given in the paper. The shift $M_1=0$ for $\varphi_0=\pm\frac{1}{2}\pi$. 2. The cylinder temperature is the sum of two harmonics so that it is not symmetric even if the fluid temperature is sinusoidal. Thus an asymmetrical response is not necessarily the result of a non-harmonic forcing function. *J. Kestin.*

Giraud, P., et Kiveliovitch, M. Quelques remarques sur les invariants en météorologie. I, II. J. Sci. Météorol. **4**, 81–86 (1952); **5**, 127–128 (1953). (English and Spanish summaries)

The authors claim to prove that an invariant, "following the motion" of a barotropic fluid, found by Ertel and Rossby is in fact not an invariant. In the course of their proof, they arrive at a formula (their eq. (4)) for the rate of change following the motion of the components of vorticity of the fluid which appears to differ from that deduced by this reviewer [see Proc. Roy. Soc. London. Ser. A. **194**, 285–300 (1949), eq. (6.8); MR **10**, 633]. The additional term found by the authors involves the divergence of the (variable) angular velocity of the coordinate axes. This reviewer has been unable to find the source of the discrepancy because the form of the equation of continuity used by them is not given. The refutation of the Ertel and Rossby result admittedly refers to the case when the additional term is zero, but the difficulty noted above makes this reviewer feel a lack of confidence in the proof.

In the second paper the authors state that they have found an error in their paper and that in their eq. (8) the absolute velocity should be used instead of the velocity relative to the rotating axes. When this is done, the result proved by Ertel and Rossby is found to be correct.

G. C. McVittie (Urbana, Ill.).

Bessemoulin, J. Confluence—diffluence. Convergence—divergence. J. Sci. Météorol. **4**, 47–60 (1952). (English and Spanish summaries)

The author shows how difficult it is in practice to calculate the divergence of the velocity of the air from the spatial rates of change of the velocity components of the wind and how easy it is to deduce that the vertical component of velocity is directed downwards. He proposes to define diffluence and confluence by considering the rate of change along the direction of motion of the air of the cross-section of a tube of flow. By this means a quantity called the diffluence of the velocity of the air can be calculated and used to find the divergence of the velocity. Brief applications to the following cases are included: special configurations of the stream-lines; the calculation of vertical velocities; the evaluation of the tendency.

G. C. McVittie (Urbana, Ill.).

Elasticity, Plasticity

Truesdell, C. The simplest rate theory of pure elasticity. Comm. Pure Appl. Math. **8**, 123–132 (1955).

This note concerns hypo-elastic bodies of grade zero, the simplest case of the general theory meanwhile developed further by the author in the paper reviewed below.

W. Noll (Los Angeles, Calif.).

Truesdell, C. Hypo-elasticity. J. Rational Mech. Anal. **4**, 83–133 (1955).

Die lineare Elastizitätstheorie kann bekanntlich prinzipiell nur näherungsweise für kleine Verzerrungen gültig sein. In dieser grundlegenden Untersuchung wird nun eine Theorie des elastischen Verhaltens entwickelt, die für beliebige endliche Verformungen gültig ist und die für kleine Verschiebungen aus einem ungespannten Zustand wieder auf die lineare Theorie führt. Es dürfte sich um die allgemeinste Theorie handeln, die dies leistet. Die Theorie der Hypo-Elastizität enthält die klassische Theorie der endlichen elastischen Deformationen als Spezialfall, ist aber weit allgemeiner und bietet einfache Modelle für das elastische Verhalten bei grossen Deformationen, die von der klassischen Theorie nicht erfasst werden. Der Ausgangspunkt der Theorie ist die anschaulich physikalische Beziehung

$$\text{Spannungsänderung}=f(\text{Verzerrungsänderung, Spannung}).$$

Es wird angenommen, dass keine Materialkonstanten auftreten, deren physikalische Dimension unabhängig von der einer Spannung ist (keine "Relaxation"). Nach einer Dimensionsanalysis ergibt sich dann unter Berücksichtigung gewisser Invarianzforderungen, dass die Materialgleichung für isotrope hypo-elastische Stoffe die folgende Form haben muss:

$$(1)\quad \tilde{\mathbf{s}}=\delta g_0\mathbf{1}+g_1\mathbf{d}+\delta g_2\mathbf{s}+Mg_3\mathbf{1}+g_4(\mathbf{ds}+\mathbf{sd})+\delta g_5\mathbf{s}^2+Mg_6\mathbf{s}+Ng_7\mathbf{1}+g_8(\mathbf{ds}^2+\mathbf{s}^2\mathbf{d})+Mg_9\mathbf{s}^2+Ng_{10}\mathbf{s}+Ng_{11}\mathbf{s}^2.$$

Dabei ist $\mathbf{s}$=Spannungstensor, $\mathbf{d}$=Deformationsgeschwindigkeitstensor, $\tilde{s}^{ij}=\dot{s}^{ij}-s^{ih}v^j{}_{,h}-s^{hj}v^i{}_{,h}+s^{ij}v^h{}_{,h}$ (v^i=Geschwind-

igkeit), $\mathbf{1}$=Einheitstensor, $\delta=\mathrm{Sp}(\mathbf{d})=d_i{}^i$, $M=s^i{}_j s^j{}_k d^k{}_i$ und $N=s^i{}_j s^j{}_k d^k{}_i$ sind simultane Invarianten von $\mathbf{s}$ und $\mathbf{d}$, die Koeffizienten $g_0, \cdots, g_{11}$ sind skalare Funktionen der drei Hauptinvarianten von $\mathbf{s}$. Hypo-elastische Stoffe haben im allgemeinen keinen natürlichen ("undeformierten") Zustand, und die Theorie gestattet somit (im Gegensatz zur klassischen Theorie) die Behandlung von Problemen mit beliebiger Anfangsspannung. Dies ist durch das Auftreten der zeitlichen Ableitung der Spannung in (1) bedingt. Im Falle homogener Spannungsverteilung beweist der Autor einen Satz, der die Existenz und Eindeutigkeit der Lösung der massgebenden Differentialgleichungen (Materialgleichung, Bewegungsgleichung und Kontinuitätsgleichung) bei gegebener Anfangsspannung und Anfangsgeschwindigkeit sichert.

Einfache spezielle hypo-elastische Stoffe sind solche, für die die rechte Seite von (1) von der Ordnung n ($=0, 1, 2, \cdots$) in den Komponenten von $\mathbf{s}$ ist. Der Autor nennt sie "hypo-elastische Körper vom Grade n". Der Fall $n=0$ wurde von ihm bereits in einer früheren Note behandelt [siehe das vorstehende Referat]. Um die Theorie der experimentellen Prüfung zugänglich zu machen, werden verschiedene spezielle Probleme behandelt: Reine Volumendehnung (hydrostatic stress), einfache Scherung, einfache Längsdehnung und Torsion eines Kreiszylinders. Die Resultate beziehen sich hauptsächlich auf hypo-elastische Stoffe vom Grad 0, 1 und 2. Je nach den Werten der zahlreichen Materialkonstanten hat man eine Vielfalt von Verhaltensweisen. Im Falle $n=2$ gibt es für die reine Volumendehnung 14 verschiedene Typen und für die einfache Scherung sogar 42. Alle diese Typen werden durch Diagramme erläutert. Es ergeben sich verschiedene Fälle von Verfestigung (strain-hardening) und Erweichung (strain-softening), und es können Erscheinungen auftreten, die auf plastisches Fliessen oder Bruch hindeuten. Wie in der klassichen Theorie der endlichen elastischen Deformationen, so lässt sich auch hier eine reine Scherung niemals durch Schubspannungen allein erzeugen; es müssen zusätzlich Normalspannungen angebracht werden ("Poynting Effekt"). Alle hier angegebenen Lösungen sind exakt und genügen auch den dynamischen Bewegungsgleichungen. Trägheitseffekte werden also nicht vernachlässigt (wie es in einem grossen Teil der Literatur sonst geschieht). An einem Beispiel wird demonstriert, wie die Vernachlässigung der Trägheitsglieder zu schweren Irrtümern führen kann.

In §11 ist übrigens eine allgemeine Lösung der dynamischen Gleichungen für die Torsion von Kreiszylindern enthalten, die auch zur Behandlung anderer Probleme der Mechanik der Kontinua von Nutzen sein dürfte.

W. Noll (Los Angeles, Calif.).

Rivlin, R. S., and Ericksen, J. L. Stress-deformation relations for isotropic materials. J. Rational Mech. Anal. **4**, 323–425 (1955).

Die Autoren untersuchen Materialgleichungen der Form

$$(1) \qquad t_{ij}=f_{ij}(x^k{}_{,\alpha}, v^{(1)}_{k,l}, v^{(2)}_{k,l}, \cdots, v^{(m)}_{k,l}),$$

wo t_{ij}=Komponenten des Spannungstensors $\mathbf{T}$, $x^k{}_{,\alpha}$=Gradient der Verschiebung $x^k=x^k(X^\alpha)$, $v^{(r)}_{k,l}$=Gradient der $(r-1)$-ten materiellen zeitlichen Ableitung der Geschwindigkeit ($v_k{}^{(1)}=v_k$=Geschwindigkeit, $v_k{}^{(2)}=\dot{v}_k$=Beschleunigung usw.). Die Forderung, dass bei isotropen Stoffen die funktionale Form der Beziehung (1) für alle beliebig bewegten Bezugssysteme die gleiche sein muss, führt zu Einschränkungen für die Funktionen f_{ij}. Es ergibt sich, dass (1) auf die Form

$$(2) \qquad \mathbf{T}=\mathfrak{F}(\mathbf{C}, \mathbf{A}_1, \cdots, \mathbf{A}_n)$$

zurückführbar sein muss, wo $\mathbf{C}$ der durch $C^{ij}=x^i{}_{,\alpha}x^j{}_{,\alpha}$ definierte Cauchysche Deformationstensor ist und wo die symmetrischen Tensoren $\mathbf{A}_r$ durch

$$A^{(r)}_{ij}=v^{(r)}_{i,j}+v^{(r)}_{j,i}+\sum_{s=1}^{r-1}\binom{r}{s} v^{(r-s)}_{k,i}\, v^{k(s)}_{\ ,j}$$

definiert sind. Ferner muss (2) eine isotrope Relation, also invariant gegenüber simultanen orthogonalen Transformationen von $\mathbf{T}$, $\mathbf{C}$, $\mathbf{A}_r$ sein. (Es sei darauf hingewiesen, dass der Spezialfall $\mathbf{T}=\mathfrak{F}(\mathbf{C}, \mathbf{A}_1)$ von (2) die korrekte dreidimensionale, nichtlineare Verallgemeinerung eines visko-elastischen Modells vom Voigt-Kelvin-Typ dastellt.)

In der zweiten Hälfte der Arbeit wird die Frage untersucht, wie man eine isotrope Relation $\mathbf{T}=\mathfrak{F}(\mathbf{A}_1, \cdots, \mathbf{A}_n)$ in eine mehr explizite Form bringen kann, und zwar dadurch, dass $\mathbf{T}$ als endliche Linearkombination einfacher Ausdrücke (z.B. $\mathbf{A}_1, \mathbf{A}_1{}^2, \mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1, \mathbf{A}_1{}^2\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1{}^2$) erscheint, wobei die Koeffizienten skalare Invarianten der $\mathbf{A}_r$ sind. Auch die Frage der Darstellung dieser Invarianten als Funktionen einer bestimmten Anzahl von Grundinvarianten wird behandelt. Es werden dabei sehr viele Fallunterscheidungen vorgenommen, die hier nicht im einzelnen besprochen werden können. Wichtiges Teilergebnis: Im Falle $n=2$ gibt es immer eine Darstellung der Form

$$\alpha\mathbf{T}+\alpha_1\mathbf{A}_1+\alpha_2\mathbf{A}_2+\alpha_3\mathbf{A}_1{}^2+\alpha_4\mathbf{A}_2{}^2+\alpha_5(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1)$$
$$+\alpha_6(\mathbf{A}_1{}^2\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1{}^2)+\alpha_7(\mathbf{A}_2{}^2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2{}^2)+\alpha_8\mathbf{I}=0.$$

Dieses Resultat wurde schon in der oben referierten Arbeit von C. Truesdell [S. 86] und in einer Arbeit vom Referenten [J. Rational Mech. Anal. **4**, 3–81 (1955), S. 27; MR **16**, 764] mit Nutzen angewandt.

(Nach Ansicht des Referenten wäre diese wichtige Arbeit kürzer und leichter lesbar geworden, hätten die Autoren mehr Gebrauch von der Sprache der Vektorräume und linearen Transformationen gemacht.) *W. Noll.*

Sonntag, G. Ein Integralsatz der ebenen Elastizitätstheorie. Z. Angew. Math. Mech. **34**, 469–471 (1954).

Proof of the theorem: "Eine unendliche Scheibe von überall gleicher Stärke sei in ihrer Ebene durch beliebige endliche Kräfte und Momente belastet; über den Weg eines beliebigen Kreises verschwindet die Spannungssumme $(\sigma_x+\sigma_y)$ des durch die eingekreiste Belastung erzeugten Spannungszustandes."

Netrebko, V. P. Torsion of an elastic parallelepiped with a given law of distribution of the shearing stresses on the bases. Vestnik Moskov. Univ. **9**, no. 12, 15–26 (1954). (Russian)

An approximate solution of the torsion problem for an elastic rectangular parallelepiped with specified distribution of shearing stresses on the bases, is obtained by minimizing the potential energy of deformation. A numerical example suggests that the stress distribution does not deviate materially from Saint-Venant's distribution in cross-sections whose distance from the loaded ends exceeds the greatest linear dimension of the ends. *I. S. Sokolnikoff.*

Sigua, F. D. Some boundary problems for a spherical shell. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 317–336 (1954). (Georgian. Russian summary)

The following problem is considered in this paper. Find the deformed state of a shell whose middle surface coincides

with the spherical segment $0 \leqq \theta \leqq \theta_0$, $\theta_0 < \pi$, and whose boundary is on a conical support or is free.

From the author's summary.

Hara, I. S. Investigation of the stress concentration under tension in infinite plates weakened by arch-shaped or trapezoidal openings. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 294–298 (1953). (Ukrainian. Russian summary)

Hara, I. S. Investigation of the stress concentration in a heavy half-plane near arch-shaped or trapezoidal openings stiffened by absolutely rigid rings. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 299–303 (1953). (Ukrainian. Russian summary)

Problems stated in the titles of these papers are solved by the Muskhelišili method with the aid of conformal mapping functions transforming the exteriors of the regions under consideration onto a unit circle. *I. S. Sokolnikoff.*

Lihačev, V. A. Remark on the paper of A. I. Kalandiya, "Bending of an elastic plate into the form of an elliptic ring". Prikl. Mat. Meh. **19**, 255–256 (1955). (Russian)

The note points out a mistatement made by A. I. Kalandiya in a footnote on p. 702 of his paper [same journal **17**, 693–704 (1953); MR **15**, 759], and indicates how certain equations in that paper can be deduced by a method of M. P. Šeremet'ev's paper [ibid. **17**, 107–113 (1953); MR **14**, 1145]. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Lehnickiĭ, S. G. The stress distribution in an anisotropic plate with an elliptic elastic core (plane problem). Inžen. Sb. **19**, 83–106 (1954). (Russian)

The author solves the problem for infinite plates using a perturbation method, based on the solution for the homogeneous infinite plate loaded uniformly at infinity. The special cases considered comprise a circular core in orthotropic plates under uniaxial and biaxial tension and shear, an elliptic core in a plate subject to stresses varying linearly with the rectilinear coordinates, a circular core in a plate in uniform bending. *J. R. M. Radok* (Providence, R. I.).

Sen Gupta, A. M. Stresses due to diametral forces on a circular disk with an eccentric hole. J. Appl. Mech. **22**, 263–266 (1955).

Cleaves, H. F. The stresses in an aeolotropic circular disk. Quart. J. Mech. Appl. Math. **8**, 59–80 (1955).

On the basis of a complex variable formulation previously employed by Livens and Morris [Phil. Mag. (7) **38**, 153–179 (1947); MR **9**, 121] the author establishes the solution to the plane problem appropriate to an aelotropic circular disk under edge tractions which admit a Fourier series representation. Numerical values are given for the special case in which the material is oak and the boundary is subjected to two equal and opposite concentrated loads applied perpendicular to the grain. *E. Sternberg* (Chicago, Ill.).

Jung, H. Druckverteilung unter elastisch gelagerten Kreisplatten. Ing.-Arch. **20**, 8–12 (1952).

The pressure distribution between an elastic symmetrically loaded circular plate and an elastic half-space is sought. In particular, the case is considered for which the contact area is a circle smaller than the plate. The deflections of the plate and the surface of the half-space are represented by integral expressions giving an integral equation for the contact condition, which is solved by means of a series expansion of Bessel functions. The contact condition is satisfied at discrete radii. An example of central point loading is treated using three terms of the series which permit contact to be satisfied at three values of the radius, and the contact radius is found to be about three tenths of the plate radius. *E. H. Lee* (Providence, R. I.).

Huth, J. H., and Cole, J. D. Impulsive loading on an elastic half-space. J. Appl. Mech. **21**, 294–295 (1954).

Eschler, H. Über freie Biegungsschwingungen des axial belasteten Stabes mit innerer und äusserer Dämpfung. Ing.-Arch. **20**, 1–5 (1952).

The equation for flexural vibrations of a rod is developed including the influence of rotary inertia, internal and external damping, and an axial force. Approximate modes are considered in which the displacement is given as the product of a function of position and a function of time. A differential equation for the function of time is obtained from Lagrange's equations and an estimated mode shape, the solution being a complex exponential which determines the natural frequency and the damping constant of the mode. An example of a damped beam is given, the assumed mode shapes being those given by simple elastic beam theory. *E. H. Lee* (Providence, R. I.).

Nomura, Yûkichi, and Takaku, Kôshun. On the propagation of elastic waves in an isotropic homogeneous sphere. J. Phys. Soc. Japan **7**, 204–211 (1952).

A steady state solution for the case when $\omega_r = \omega_\theta = 0$ and all functions depend on r and θ only is used to express the dilatation and the third component of rotation in terms of constant coefficients of reflection. Both primary P and S diverging waves are considered. It is assumed they are reflected at the surface of the sphere ($r = a$) and also at an infinitesimal rigid sphere at the center of the elastic sphere. Then, the expressions for multiply reflected P and S waves are built up. These expressions are transformed by the method of van der Pol and Bremmer [Phil. Mag. (7) **24**, 141–176, 825–864 (1937)]. *PP*, *PS*, *SP*, *SS*-waves as well as surface waves are derived. *W. S. Jardetzky.*

Selberg, Henrik L. Transient compression waves from spherical and cylindrical cavities. Ark. Fys. **5**, 97–108 (1952).

The radial displacement u_r in an infinite solid can be expressed in terms of a function $\psi(t-(r-r_0)/c)$ and its derivative. The boundary condition at the cavity surface $r = r_0$ yields a differential equation of the second order for this function. For $t > 0$ the normal pressure $P(t)$ is assumed to be uniformly distributed over $r = r_0$. If $P(t) = 0$ for $t = t_1$ the solution is a damped vibration. Curves representing radial, tangential and maximum shear stresses were calculated for $P(t) = -1$. In a similar treatment of waves originating at a cylindrical cavity, use has been made of the Laplace transform of the dilatation. *W. S. Jardetzky* (New York, N. Y.).

Muštari, H. M., and Sačenkov, A. V. On stability of cylindrical and conical shells of circular section under the simultaneous action of axial compression and external normal pressure. Prikl. Mat. Meh. **18**, 667–674 (1954). (Russian)

The authors apply Galerkin's method to the solution of the equilibrium equations for conical shells and deduce

certain approximate formulae for the critical external pressure in the presence of axial compression.

J. R. M. Radok (Providence, R. I.).

Leggett, D. M. A. The buckling of thin cylindrical shells under axial compression. Acad. Serbe Sci. Publ. Inst. Math. **7**, 47–60 (1954).

The author extends the theory of Karman and Tsien by minimizing the strain energy with respect to four parameters instead of just two. The resulting load-overall strain curve may contain discontinuities of the load where the energies of two possible modes of distortion, corresponding to different numbers of circumferential buckles, are equal: the mode actually adopted being that with the least energy at each strain point. *D. R. Bland* (London).

Benthem, J. P. On the stress-strain relations of plastic deformation. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **S. 398**, iii+88 pp. (19 plates) (1951).

This survey report comprises a critical discussion of the mathematical formulations which have been proposed governing the relationship between stress and strain in plastic flow. The emphasis is on rate-independent laws. The development and structure of deformation-type laws, flow-type laws, and the slip theory are discussed, and also agreement with experimental measurements. The report commences with the derivation of strain and stress representations and a discussion of tensor invariants. *E. H. Lee.*

Fastov, N. S. On the equations of the theory of plasticity taking account of temperature variation. Dokl. Akad. Nauk SSSR. (N.S.) **85**, 67–70 (1952). (Russian)

In the presence of temperature change, the ratio of the deviators of stress and strain, which in the total strain law of Ilyušin [Plasticity, Part I, OGIZ, Moscow-Leningrad, 1948; MR **12**, 373] is normally a function only of the invariants of strain, becomes also a function of the temperature. When the total strain remains small, the appropriate function for a perfectly elastic, linearly strain-hardening material is shown to include an extra term depending linearly on the temperature change. *R. M. Haythornthwaite.*

Ericksen, J. L. Singular surfaces in plasticity. J. Math. Phys. **34**, 74–79 (1955).

The author investigates the conditions under which surfaces of weak velocity discontinuity can arise in the three-dimensional problem of the v. Mises theory of perfectly plastic solids. Thomas [J. Rational Mech. Anal. **2**, 339–381 (1953); MR **14**, 820] proved that velocity discontinuities of the order two are subject to severe restrictions and pointed out the possibility that discontinuities of the order one might occur under less restrictive conditions. The present paper shows that this is not the case. In particular, it is shown that the velocity strain corresponds to simple shear at each point of a first-order discontinuity surface of the velocity field, the plane of shear containing the normal of the discontinuity surface. This implies that the discontinuity surface moves instantaneously as though it were an inextensible membrane. *W. Prager* (Providence, R. I.).

Nabarro, F. R. N. The mathematical theory of stationary dislocations. Advances in Physics **1**, 269–394 (1 plate) (1952).

Vorliegende Arbeit ist eine Zusammenfassung unserer heutigen sich auf die Theorie der stationären Versetzungen beziehenden Kenntnisse. Nach einem historischen Überblick über die Entwicklung dieser Theorie, die ihren Ausgangspunkt vor einem halben Jahrhundert noch aus solchen Vorstellungen gehabt hat, dass Elektronen "Versetzungen des Aethers" usw. sind, wird zuerst die elementare Theorie der Versetzungen besprochen; die Begriffe der Stufen- und der Schraubenversetzungen werden eingeführt und die Grundzüge der Burgersschen Theorie werden dargestellt. Im darauffolgenden Teil wird die mathematische Theorie der Bewegung von Versetzungen besprochen; die Frank-Readsche Quelle von Versetzungen, die auf eine Versetzung einwirkende Kraft und die Kräfte welche Versetzungen aufeinander ausüben, werden behandelt. Danach wird das Problem der gehäuften Versetzungen und im Zusammenhang damit die Theorie der Verfestigung, ausserdem die Wechselwirkungen von Versetzungen mit freien Oberflächen und die Energien der Versetzungen besprochen. Der letzte Abschnitt behandelt endlich den Einfluss der Kristallstruktur auf die Versetzungen.

In der ganzen bisherigen Theorie der Versetzungen benützt man das Verfahren, dass man dort wo die Versetzungen tatsächlich auftreten, aus der Gitterstruktur der festen Materie ausgeht, in deren Umgebungen dagegen die klassische Elastizitätstheorie benutzt. Interessant ist es jedoch zu bemerken, dass man die modernen Resultate der Quantenmechanik, bezüglich der Kohäsion der festen Materie, in diese Theorie noch nicht eingeführt hat.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Laugwitz, Detlef. Zur Rolle der pythagoreischen Metrik in der Physik. Z. Naturf. **9a**, 827–832 (1954).

The author introduces a characterisation of (not necessarily finite-dimensional) spaces with a Pythagorean metric by means of an isotropy postulate. This has the advantage that the configuration spaces of quantum mechanics may be included under this characterisation as well as the euclidean and riemannian spaces of classical and relativistic physics. The isotropy postulate appears to be the most essential aspect of the author's theory since it implies that there no preferred states with respect to observation.

H. Rund (Toronto, Ont.).

Watanabe, Satosi. Symmetry of physical laws. II. q-number theory of space-time inversions and charge conjugation. Rev. Modern Phys. **27**, 40–76 (1955).

The paper is devoted to a detailed discussion of the covariance of the most important field theories for space-inversion, time-reversal and charge conjugation. After giving an account of the main properties of spinors (after the method of E. Cartan based on the decomposition of each Lorentz transformation in a product of reflections), the author studies the unitary operators corresponding to space-inversion, time-reversal and charge conjugation for the fields of common interest. [For part I see MR **16**, 890.]

L. Van Hove (Utrecht).

Optics, Electromagnetic Theory

*Gramatzki, H. J. **Probleme der konstruktiven Optik und ihre mathematischen Hilfsmittel.** Akademie-Verlag, Berlin, 1954. vii+140 pp.

The book contains a series of very well presented articles on special optical problems, mostly of an elementary nature. The subject matter is well organized, thus making the little volume a helpful tool for the optical designer. We mention from the table of contents: Formulae for ray-tracing for finite aperture and in the paraxial region, graphical methods of ray-tracing, color correction, and special lenses, in which, among other things, the Mangin mirror and the Schmidt correction plate are discussed. *M. Herzberger.*

Gal'pern, D. Yu. Geometrical optics in anisotropic media with a single axis of symmetry. (Uniaxial crystals.) Ž. Tehn. Fiz. **24**, 1298–1321 (1954). (Russian)

A method based on Fermat's principle is described for finding the path of a ray through an optical system containing uniaxial crystals. The properties of the derivatives of the angular, mixed, and Schwarzschild eiconals are investigated at some length. These basic results are applied to the study of the behavior of infinitely narrow meridional and sagittal pencils in systems with a meridional plane. For this problem the connection is shown between focal distances, sagittal as well as meridional, in the object and image spaces, and formulas are derived analogous to Newton's formulas for linear and angular magnification. The basic formulas are also used to derive the laws of reflection and refraction at the boundary between two uniaxial crystals. The results obtained are greatly simplified if the incident ray, the normal at the point of incidence, and the axis of the first crystal are coplanar. If the axis of the second crystal is also coplanar with these, the study of the astigmatism of infinitely narrow pencils becomes possible without any excessive complexity in the formulas. Applications of the general method are given to the case of prisms and to the determination of the paths of paraxial rays through a system of surfaces having an axis of symmetry when the axes of the crystals are parallel to each other and perpendicular to the axis of the system. *J. E. Rosenthal* (Passaic, N. J.).

Reuschel, A. Zur kinematischen Behandlung optischer Probleme. Optik **12**, 128–135 (1955).

Blanc-Lapierre, André, et Dumontet, Pierre. La notion de cohérence en optique. Rev. Opt. **34**, 1–21 (1955).

The problem of coherence and partial coherence in optics is studied as an application of correlation theory. The author studies coherence coming from the fact that not monochromatic light but a narrow spectral band is used to illuminate the object as well as the coherence of the optical image due to diffraction or interference. This is one of the important papers coordinating different fields of science which are separated mostly by the use of different languages developed in each of them. An appendix compares the statistical qualities of light with the Gauss-Laplace formulas used for statistical problems. *M. Herzberger.*

Fellgett, P. B., and Linfoot, E. H. On the assessment of optical images. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 369–407 (1955).

This paper contains an excellent introduction to the modern ideas of information theory as applied to optics.

The author gives the mathematical background of the Fourier transformations describing the object-image connection in optical systems. It is significant that he distinguishes between image fidelity in optical systems in which the object is more or less sharply imaged and image assessment in which information is extracted by a reconstruction process.

The author ties up the optical information with the optical characteristic function, applying to it the methods of information theory and separating "noise" from information. The reviewer is somewhat in doubt whether the specific evaluation given by information theory does take care of the optically important distinction of a symmetric and asymmetric nucleus. This can be easily taken care of if, in the evaluation, the phase and amplitude are both considered. *M. Herzberger* (Rochester, N. Y.).

Krieger, Rudolf. Über die Lichtverteilung im Bild eines Linienstückchens bei Aberrationsfunktionen 2. Grades in den Pupillenvariablen. Opt. Acta **1**, 9–20 (1954).

Krieger, Rudolf. Über die Lichtverteilung im Bild eines Linienstückchens bei Aberrationsfunktionen 3. Grades in den Pupillenvariablen. Mitt. Math. Sem. Giessen no. **46**, i+36 pp. (9 plates) (1954).

The author discusses the intensity variation of the light distribution in the image of sagittal and meridional line elements considering the aberrations near a principal ray up to the third order. The paper contains a large number of interesting graphs coordinating aberration curves and intensity distribution curves. *M. Herzberger.*

Vandakurov, Yu. V. Diffraction by a perfectly conducting half-plane of electromagnetic waves emanating from an arbitrarily oriented electric or magnetic dipole. Ž. Eksper. Teoret. Fiz. **26**, 3–18 (1954). (Russian)

The author solves the problem by using two scalar potentials (solutions of the wave equation with a point source, with either the function or its normal derivative vanishing on the half-plane). He does not derive all field components from a single potential, however, thus avoiding solutions with too high singularities at the edge; for each field component the appropriate singularity and boundary conditions are used. A proof of uniqueness following Meixner [Ann. Physik (6) **6**, 2–9 (1949); MR **11**, 562] is included. The author is apparently unaware of the alternative method used by Senior [Quart. J. Mech. Appl. Math. **6**, 101–114 (1953); MR **14**, 933] in a paper published just before this paper was submitted for publication. *J. Shmoys.*

Heins, A. E., and Silver, S. The edge conditions and field representation theorems in the theory of electromagnetic diffraction. Proc. Cambridge Philos. Soc. **51**, 149–161 (1955).

The paper deals with the case of a finite aperture bounded by a smooth closed curve in a plane perfectly conducting screen of otherwise infinite extent. After a brief survey of the history of the edge conditions, the authors first develop order conditions which must be satisfied by the field components in the neighbourhood of the edge as a consequence of the requirement that the total energy in a finite volume must be finite. These order conditions are then regarded as part of the specification of the boundary-value problem. Secondly, having a knowledge of these order conditions, the authors formulate the boundary-value problem as a pair of simultaneous integral equations which are solved in the

neighbourhood of the edge. In this way they determine the functional form of the field components in the region local to the edge without the assumption of a particular type of expansion [cf. J. Meixner, Ann. Physik (6) 6, 2–9 (1949); MR 11, 562]. *C. J. Bouwkamp* (Eindhoven).

Meixner, Josef. The behavior of electromagnetic fields at edges. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-72**, i+12 pp. (1954).

In diffraction problems, the electromagnetic field vectors may become infinite at a sharp edge of the diffracting obstacle, the order of the singularity being subject to the condition that the electromagnetic energy density must be integrable over any finite domain, even if this domain contains singularities. In the case of a perfectly conducting surface with one edge the singular components of the electric and magnetic vectors are of order $\rho^{-1/2}$ where ρ is the distance from the edge, while the components parallel to the edge are alwa$_y$s finite. In this paper, the above result is generalised to find the behaviour of the field vectors near the edges of the dielectric and perfectly conducting bodies. Actually the author considers the particular case of a space filled with wedges of homogeneous material with a common straight edge: in particular he discusses two cases: (i) two dielectric wedges and one perfectly conducting wedge; (ii) two dielectric wedges. *E. T. Copson* (St. Andrews).

Karp, S. N. The effect of discontinuities of dielectric contant on electrostatic fields near conductors. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-71**, i+34 pp. (1954).

The author determines the behaviour of an electrostatic field near the vertex of a wedge-shaped conductor in the presence of a medium whose dielectric constant has a plane surface of discontinuity passing through the edge of the wedge and is constant on each side of this plane. The field becomes singular as the edge is approached, and is of the form $r^{\lambda-1}$ where r is the distance from the edge and λ depends on the two values of the dielectric constant and also upon the angles involved. It turns out that if the medium with greater dielectric constant occupies a greater angular region than the other, the order of the singularity at the edge of the wedge is greater than the order if the medium were homogeneous. If both media occupy equal angular regions, then the order is unaltered. If the medium with greater dielectric constant occupies the smaller angle, the order is decreased.

The Green's function for such a problem is considered. When the wedge is symmetric with respect to the interface, the Green's function can be expressed by the method of images in terms of the Green's function when the medium is homogeneous. This last result is extended to the case of an arbitrary symmetric distribution of conductors.

E. T. Copson (St. Andrews).

Goudal, Pierre. Sur la théorie des bobines de Helmholtz. Rev. Gén. Elec. **39**, 199–202 (1955).

L'auteur expose, dans cet article, une étude des champs créés par des bobines de Helmholtz en un point de leur axe, à partir du potentiel magnétique correspondant. Ce potentiel en un point est lui-même calculé préalablement dans le cas de deux courants circulaires en utilisant un développement en série de polynômes de Legendre.

Résumé de l'auteur.

***Andreev, N. N. The method of averaging in the solution of wave problems.** Sbornik posvyaščennyi semidesyatiletiyu akademika A. F. Ioffe [Collection in honor of the seventieth birthday of academician A. F. Ioffe], pp. 467–473. Izdat. Akad. Nauk SSSR, Moscow, 1950.

The paper deals with propagation in waveguides of arbitrary, constant or varying cross-section. Since for constant cross-section, in the case of normal derivation vanishing on the boundary, the average of the wave function over the cross-section satisfies the wave equation in the longitudinal direction, the author believes it might be fruitful to use this average in other cases. (Since, however, the method of separation of variables is just as simple and it does not automatically eliminate higher-order modes, the reviewer finds it difficult to agree with the author.) This approximate method is then applied to two specific problems which could have been dealt with just as easily and with more rigor by the method of separation of variables. *J. Shmoys.*

Marcuvitz, Nathan. Field representations in general cylindrical regions. I. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-69**, i+26 pp. (1954).

In the present paper, the first of a series of three, the author extends the standard treatment of the vector modes in a cylindrical waveguide [see, e.g., J. C. Slater, Microwave electronics, Van Nostrand, New York, 1950] to a cylindrical structure with arbitrary ϵ and μ variation, and an arbitrary boundary condition. Various orthogonality properties are demonstrated for the vector modes, and a completeness relation is stated. The proof of the latter is deferred to a later section. Emphasis is placed on the reduction of Maxwell's equations to transmission-line equations, and the reformulation of the electromagnetic problem as a network problem. The paper closes with a section devoted to planar stratified regions, especially those with all parameters held uniform in both directions perpendicular to the stratifications. *W. K. Saunders* (Washington, D. C.).

Jancel, R., et Kahan, T. Théorie du couplage des ondes électromagnétiques ordinaire et extraordinaire dans un plasma inhomogène et anisotrope et conditions de réflexion. Applications à l'ionosphère. J. Phys. Radium (8) **16**, 136–145 (1955).

Les auteurs rappellent rapidement les résultats de leur travaux antérieurs sur la conductivité des plasmas anisotropes [C. R. Acad. Sci. Paris **236**, 788–790, 1478–1481, 2045–2047; **237**, 1657–1659 (1953); **238**, 995–996 (1954); Phys. Radium (8) **14**, 533–540 (1953); **15**, 26–33 (1954); MR **15**, 665]. Ils considèrent un champ électrique oscillant et un champ magnétique constant. En utilisant la méthode d'approximations B. K. W., ils obtiennent pour la force électrique $\mathcal{E}$ une expression ou intervient une grandeur complexe $Y_{\pm}$ qui caractérise l'indice de réfraction (et d'absorption) pour le rayon ordinaire et pour le rayon extraordinaire. Le but principal de l'article présent est l'étude de la fonction Y de l'altitude z sur sa surface de Riemann, surface à quatre feuillets dont deux correspondent au rayon ordinaire et deux au rayon extraordinaire. Les résultats sont applicables à l'ionosphère. *H. Bremekamp* (Delft).

Kuznecov, P. I., and Stratonovič, R. L. Electromagnetic processes in a multiconductor system. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 9, 3–23 (1954). (Russian)

The authors derive, starting from Maxwell's equations, the transmission-line equations for a system of thin, parallel,

imperfectly conducting cylindrical wires. Only those modes which in the limit of perfect conductivity become transverse electromagnetic are considered. Errors introduced by assuming the wires to be thin are of the order of the square of the ratio of thickness to wavelength or to distance between wires, whichever is smaller. The method is then applied to the case of a parallel wire line over a perfectly conducting ground. The two modes of propagation of such a line are investigated in detail. *J. Shmoys* (New York, N. Y.).

Elliott, Robert S. Azimuthal surface waves on circular cylinders. J. Appl. Phys. **26**, 368–376 (1955).

Solutions of Maxwell's equations are presented which satisfy the boundary conditions for corrugated and dielectric-clad circular conducting cylinders.

From the author's summary.

Wait, James R. Field produced by an arbitrary slot on an elliptic cylinder. J. Appl. Phys. **26**, 458–463 (1955).

A derivation is given for the electromagnetic field produced by an arbitrary slot cut in the surface of an elliptic cylinder of perfect conductivity and infinite length. The electric field tangential to the slot is assumed to be a prescribed function. The integrals in the formal solution are evaluated by the saddle-point method for the far-zone fields which when specialized to thin transverse or axial slots agree with known results. As illustrative examples, patterns of thin axial and transverse slots on a strip are computed.

Author's summary.

Baños, Alfredo, Jr., and Golden, Robert K. The electromagnetic field of a rotating uniformly magnetized sphere. J. Appl. Phys. **23**, 1294–1299 (1952).

Das elektromagnetische Feld einer sich mit konstanter Winkelgeschwindigkeit drehenden homogen magnetisierten Kugel wird für die Fälle untersucht, dass erstens die Kugel allein vorhanden ist und zweitens dass sie von einer konzentrischen leitenden (jedoch nicht magnetischen) Kugelschale umgeben ist. Die Verfasser gehen aus der Minkowskischen Elektrodynamik der bewegten Körper aus und benützen das MKS-System.

In einem Koordinatensystem in dem die Kugel ruht, treten selbstverständlich nur die magnetischen Vektoren auf, alle elektrischen sind dagegen gleich Null. Danach wird ein Inertialsystem für den mit der Kugel mitbewegten Beobachter eingeführt und die elektrischen und magnetischen Vektoren werden in dieses System transformiert. Bei Vernachlässigung von v^2/c^2 neben eins folgt aus dieser Berechnung, dass sich die magnetischen Vektoren nicht ändern, was man anschaulich so ausdrücken könnte, dass die magnetischen Kraftlinien an der Bewegung der Kugel nicht teilnehmen. Bezüglich des elektrischen Feldes sind die Verhältnisse jedoch anders; im erwähnten Inertialsystem treten endliche Werte der Feldintensität, der elektrischen Verschiebung und der Polarisation im Inneren der Kugel auf, alle diese Vektoren stehen senkrecht zur Rotationsachse. Im Aussenraum tritt ebenfalls ein elektrisches Feld auf, das man als Feld eines axialen Quadrupols beschreiben kann and dessen Potential

$$V'' = -\tfrac{1}{3}\omega a^5 B^- \frac{P_2(\cos\vartheta)}{r^3}$$

ist. ω bedeutet hier die Winkelgeschwindigkeit und a den Radius der Kugel, B^- ist die magnetische Induktion und P_2 die zweite Kugelfunktion.

Alle diese Resultate beziehen sich auf den Fall, dass die Kugel allein vorhanden ist. Wird die jetzt von von einer leitenden und von der Kugel isolierten ruhenden Kugelschale umgeben, so ändern sich die Verhältnisse innerhalb der Kugel nicht. An der inneren Oberfläche der Schale werden jedoch elektrische Ladungen erscheinen und die Oberflächenladung der Kugel ändert sich; es tritt jedoch weder innerhalb des Materials dieser Kugelschale noch ausserhalb von der ein elektrisches Feld (also auch keine induzierten Ströme) auf. *T. Neugebauer* (Budapest).

Kirko, I. M. A symmetric form of Maxwell's equations according to Arkad'ev and some generalizations of the Umov-Poynting theorem. Latvijas PSR Zinātņu Akad. Vēstis **1952**, no. 7(60), 119–121 (1952). (Russian)

Maxwell's equations for a sinusoidal field and expressions for the complex Poynting vector are extended to cover the case where magnetic conductivity is present in a medium. The results are a trivial modification of the material given in any standard text on electromagnetic theory [e.g. J. A. Stratton, Electromagnetic theory, 1st ed., McGraw-Hill, New York, 1941, p. 137]. *J. E. Rosenthal.*

Furutsu, Koichi. On the group velocity, wave path and their relations to the Poynting vector of the electromagnetic field in an absorbing medium. J. Phys. Soc. Japan **7**, 458–466 (1952).

This paper examines the physical meaning and the mathematical formulation of concepts such as group velocity and wave path and considers their extension to an absorbing non-uniform medium. The mathematical treatment is based on the Kemmer formulation of the vector field (to which unfortunately no reference is given, nor could any be found in a number of standard American texts on electromagnetic theory). The following conclusions are reached: In a rigorous sense neither group velocity nor wave path has a meaning in an absorbing non-uniform medium, but where the loss is small, they agree to a first approximation with the quantities obtained in the absence of absorption; it is therefore meaningless to discuss changes in these quantities due to energy loss. These conclusions appear to be at least partly due to an attempt to ascribe too precise a physical significance to the terms in question. [Cf. J. A. Stratton, Electromagnetic theory, McGraw-Hill, New York, 1941, sec. 5.17, p. 330.]

J. E. Rosenthal (Passaic, N. J.).

Moon, Parry, and Spencer, Domina Eberle. A postulational approach to electromagnetism. J. Franklin Inst. **259**, 293–305 (1955).

Brodin, Jean. Réseaux linéaires à paramètres localisés dépendant du temps. C. R. Acad. Sci. Paris **233**, 1168–1170 (1951).

Ecriture formelle des équations d'un système au moyen d'un symbole d'intégration: application au calcul de la bande passante d'un multiplieur à réseaux dépendant du temps. *Résumé de l'auteur.*

Quantum Mechanics

Burton, W. K. Canonical invariance in Lagrangian quantum mechanics. Phys. Rev. (2) **98**, 555–556 (1955).

Franke, Herbert W. Ein Strömungsmodell der Wellenmechanik. Acta Phys. Acad. Sci. Hungar. **4**, 163–172 (1954). (Russian summary)

This gives a discussion of some aspects of a hydrodynamical, deterministic model of non-relativistic wave-mechanics. *C. Strachan* (Aberdeen).

Kristensen, P. Configuration space representation for non-linear fields. Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 12, 53 pp. (1954).

Consider a quantum field theory in which the field operators ψ, $\bar{\psi}$ and u are coupled to external source densities $\bar{\phi}$, ϕ and I respectively, and have also a non-linear interaction with one another. Define the operator T by

$$T=\exp\left[-i\int_{-\infty}^{\infty}(\bar{\psi}\phi+\bar{\phi}\psi+Iu)dx\right],$$

where the operators $\bar{\psi}$, ψ, u are Heisenberg operators and the exponential is to be time-ordered in the anti-chronological sense. For any state Ψ of the interacting fields, the matrix element $\langle 0|T|\Psi\rangle$ is a functional of the functions $\bar{\phi}$, ϕ, I, and provides a "configuration-space wave-function" for the state Ψ. Taking functional derivatives of $\langle 0|T|\Psi\rangle$ and setting $\bar{\phi}$, ϕ, $I=0$ after differentiation, one generates the matrix elements

$$\langle 0|T(u(x_1)\cdots u(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_m)\psi(z_1)\cdots\psi(z_l)|\Psi\rangle,$$

where now T represents the chronological ordering operator of G. C. Wick [Phys. Rev. (2) **80**, 268–272 (1950); MR **12**, 380].

This paper consists mainly of an expository survey of the formal definitions of Green's functions and wave-functions in quantum field theory, using the operator T defined above as starting-point. In this way the formal connections between the various definitions are derived very simply, but no essentially new results are obtained. *F. J. Dyson.*

Umezawa, Hiroomi, Kamefuchi, Susumu, and Tanaka, Shô. On the time reversal in the quantized field theory. Progr. Theoret. Phys. **12**, 383–400 (1954).

The authors enumerate and classify the possible transformations which go by the name of "time-reversal" in quantum field theory. There are two essentially distinct transformations, one introduced by E. P. Wigner [Nachr. Ges. Wiss. Göttingen **1932**, 546–559] and the other by W. Pauli [Rev. Mod. Phys. **13**, 203–232 (1941)]. The difference between the two is the operation of charge-conjugation which exchanges particle with anti-particle states. There is no a priori reason to consider either transformation a "simple time-reversal" and the other as "time-reversal plus charge-conjugation". It is only a question of convenience which transformation is taken to be primitive. In a final section it is observed that the definition of time-reversal becomes even more ambiguous when the theory is invariant under other internal transformations besides charge-conjugation, for example when the theory has invariance under the full rotation-group in isotropic space.

F. J. Dyson (Princeton, N. J.).

Hayashi, Chushiro, and Munakata, Yasuo. On a relativistic integral equation for bound states. Progr. Theoret. Phys. **7**, 481–516 (1952).

Following Feynman's methods [Phys. Rev. (2) **76**, 769–789 (1949); MR **11**, 765], a relativistic integral equation for bound states is derived. This is the same equation which was independently discovered by Salpeter and Bethe [ibid. **84**, 1232–1242 (1951); MR **14**, 707]. For the general relativistic case the motion of the centre of mass and the angular variables in the relative coordinates can be separated, leaving a Fredholm equation in two dimensions (relative distance r and time t). The case of a charged scalar field interacting through a neutral field is first investigated for the ladder approximation [cf. G. C. Wick, ibid. **96**, 1124–1134 (1954); MR **16**, 655]. If in the kernel δ_+ is replaced by $\delta(r-t)/r$, the integral equation can be explicitly solved. The eigenvalues fully correspond to those obtained by a non-relativistic treatment. Next is considered the spinor case. The singularity of the kernel is noted and a thorough discussion of the difficulties given. *A. Salam.*

Klepikov, N. P. On the theory of the vacuum functional. Dokl. Akad. Nauk SSSR (N.S.) **98**, 937–940 (1954). (Russian)

Following the ideas of J. Schwinger [Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951); MR **13**, 520], the author derives the functional differential equations satisfied by the vacuum functional G_0 of a quantized field theory. G_0 is defined as the vacuum-to-vacuum matrix element of the S-matrix in the presence of external sources which are unknown functions of position in space-time. The equations are equivalent to those used by Schwinger.

F. J. Dyson (Princeton, N. J.).

Ivanenko, D. D., and Kurdgelaidze, D. F. The basic equations of mesodynamics. Dokl. Akad. Nauk SSSR (N.S.) **96**, 39–42 (1954). (Russian)

The first part of this paper gives detailed examples of non-linear terms in the Lagrangian arising from fourth-order Feynman processes. The second, gives closed solutions of the equation $[\Box-k^2-\lambda\varphi^2]\varphi=0$ in terms of elliptic functions. This equation is "basic" in the sense that it is the simplest equation manifesting the type of non-linearity shown in §1 to be characteristic of meson theory. Finally, approximate methods of solving the equation

$$\Delta\varphi+\sum_0^\infty\lambda_s\varphi^{2s+1}=\rho(r)$$

are sketched for the cases (i) $|\Delta\varphi^n|\ll|\lambda_0\varphi^n|$ and (ii) $\lambda_s\varphi^{2s}<\lambda_0$ for $s>0$. *A. J. Coleman* (Toronto, Ont.).

Majumdar, R. C., Gupta, S., and Trehan, S. K. The classical equations of a point particle in a symmetric meson field. Progr. Theoret. Phys. **12**, 31–48 (1954).

The Wheeler-Feynman action-at-a-distance theory [Rev. Mod. Phys. **17**, 157–181 (1945); **21**, 425–433 (1949); MR **11**, 293] is extended to problems of meson scattering by point nucleons with charge symmetric theory. Cross-sections for the scattering of vector and scalar mesons are obtained and compared with those obtained by Fierz [Helv. Phys. Acta **14**, 257–270 (1941); MR **3**, 159] and others on the basis of retarded classical field. The authors find that the terms in the cross-sections, due to inertial reactions of the meson field resulting in an apparent change of mass, spin and isotopic spin of the nucleon in the retarded theory, are absent from the present theory in which the average of the advanced and retarded potentials is used.

A. J. Coleman (Toronto, Ont.).

Havas, Peter. The classical equations of motion of point particles. II. Phys. Rev. (2) **91**, 997–1007 (1953).

[For part I see Phys. Rev. (2) **87**, 309–318 (1952); MR **14**, 435.] The classical equations of motion of particles

interacting through a charge-symmetrical scalar or vector meson field are discussed along the lines of the classical formulation of electrodynamics by Dirac [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)] and of the Wheeler-Feynman theory of action at a distance [Rev. Mod. Phys. **17**, 157–181 (1945)]. The results obtained are the natural extension of those derived previously by the author for neutral meson theories [Phys. Rev. (2) **87**, 309–318 (1952); MR **14**, 435]. *L. Van Hove* (Utrecht).

Galanin, A. D. Some remarks on divergences in the theory of a pseudoscalar meson with pseudovector coupling. Ž. Eksper. Teoret. Fiz. **26**, 423–429 (1954). (Russian)

A very detailed examination of the theory of pseudoscalar meson with pseudovector coupling shows that the usual perturbation approach is inappropriate for all values of the coupling constant and energy. *A. J. Coleman.*

Makiej, Bolesław. Calculation of the π-meson mass based on an electrodynamical model of the particle. Acta Phys. Polon. **11**, 87–89 (1951).

The dispersion formula for electromagnetic waves in plasma is known to have the following form

$$u=c(1-\omega_0^2/\omega^2)^{-1/2}$$

where u is the phase velocity, ω the angular frequency of the electromagnetic wave and ω_0, the plasma frequency [D. Bohm and E. P. Gross, Phys. Rev. (2) **75**, 1851–1876 (1949)]. The analogy with the deBroglie formula suggests an electrodynamical model of a free particle, leading to an electromagnetic mass $2\hbar cm_0/e^2$ for a charged spinless particle. Here m_0 is the mass of an electron. The excellent agreement of this value with the π-meson mass is noted!

A. Salam (Cambridge, England).

Klimontovič, Yu. L. Second quantization in phase space. Dokl. Akad. Nauk SSSR (N.S.) **96**, 43–46 (1954). (Russian)

The author considers a system of particles with two-body interaction depending on distance only and obtains formulas connecting the energy and momentum in excited states for bosons and fermions, one of which, generalizes a result of Tomonaga [Progr. Theoret. Phys. **5**, 544–569 (1950); MR **13**, 414]. *A. J. Coleman* (Toronto, Ont.).

Kikuta, Takashi. Upper and lower bounds of Born approximation. I. Progr. Theoret. Phys. **12**, 225–233 (1954).

The convergence of the one-dimensional Born expansion is investigated as a function of the parameter λ measuring the strength of the scattering potential. If λ_c be the radius of convergence and $u_l(n)$ is the radial part of the wave-function for angular momentum l, it is shown that, if $u_l^{(n)}$ be the nth Born approximation and $\Delta u_l^{(n)}=u_l-u_l^{(n)}$, then $|\Delta u_l^{(n)}|/|u_l| \leqq (|\lambda|/\lambda_{l,c})^{n+1}$. (The norm of any function Ψ is defined by $|\Psi|=(\int_0^\infty \Psi^* V\Psi dr)^{1/2}$, where it is assumed that $V \geqslant 0$.) For the scattering amplitudes

$$\omega_l=\lambda k^{-1}\int_0^\infty j_l(kr)\,V(r)u_l(r)dr \qquad (j_l(kr)=(\pi kr/2)^{1/2}J_{l+\frac{1}{2}}(kr)),$$

it is shown, for example, that, for the first Born approximation,

$$-1/(\mu_l+1)<\Delta\omega_l^{(1)}/\omega_l^{(1)}<1/(\mu_l+1), \quad (\mu_l=\lambda_{l,c}/|\lambda|)$$

if $\mu_l>1$, and, for the second Born approximation,

$$0<\Delta\omega_l^{(2)}/\omega_l^{(1)}<1/\mu_l(\mu_l-1).$$

Higher Born approximations are discussed and a result based on Schwinger's variational method [Lippmann and Schwinger, Phys. Rev. (2) **79**, 469–480 (1950); MR **12**, 570] is shown to be superior to the second Born approximation. *C. Strachan* (Aberdeen).

Kikuta, Takashi. Upper and lower bounds of Born approximation. II. Progr. Theoret. Phys. **12**, 234–240 (1954).

The three-dimensional Born approximation is investigated in a manner similar to that of the paper reviewed above. A square-well potential is discussed as an example.

C. Strachan (Aberdeen).

Dicke, R. H. Angular momentum of a real field. Phys. Rev. (2) **97**, 536–539 (1955).

Angular momentum is discussed for a particular scalar field (and by implication for certain transverse and longitudinal vector fields derived from it) in order to discuss apparent conflicts between non-zero angular momenta and intuitive symmetries. *C. Strachan* (Aberdeen).

Hanus, Wanda. On the β-formalism of Kemmer and its quantization on the basis of Schwinger's variational principle. Acta Phys. Polon. **13**, 275–289 (1954). (Russian summary).

Schwinger's methods [Phys. Rev. (2) **82**, 914–927 (1951); **91**, 713–728 (1953); MR **13**, 520; **15**, 81] are applied to the problem of quantising the integer-spin fields in Kemmer's β-formalism [Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); MR **1**, 95]. The problem is complicated by the field variable having to be divided into dynamically independent ("canonical") and dependent ("constraint") parts. The author notes that "the quite correct" procedure should be to follow Schwinger's work in paper II cited above. He however limits himself to the simplified formalism based on paper I, thus obtaining the commutation relations without the factor 1/2. The necessity of this factor for the consistency of the commutation relations was noticed not long ago by Burton and Touschek [Phil. Mag. (7) **44**, 161–168 (1953); MR **14**, 1045] and Schwinger [ibid. **44**, 1171–1179 (1953); MR **15**, 764]. The author's results are the same as those of Kemmer which also lack this factor.

A. Salam (Cambridge, England).

Sokolik, G. A. Remarks on the theory of fusion. Ž. Eksper. Teoret. Fiz. **28**, 13–16 (1955). (Russian)

Let two representations τ^1 and τ^2 of the Lorentz group be given, which may be reducible or irreducible, finite- or infinite-dimensional. The direct product $(\tau^1\times\tau^2)$ is here decomposed by the methods of I. Gel'fand and A. Yaglom [same Ž. **18**, 703–733 (1948); MR **10**, 583] into a direct sum of representations τ_m. To each τ_m corresponds a "particle" which satisfies a wave-equation which can be constructed explicitly from a knowledge of the structure of τ^1 and τ^2. This method of constructing wave-equations is a generalization of the "method of fusion" of L. de Broglie [Théorie générale des particles à spin, Gauthier-Villars, Paris, 1943]. The discussion in this paper is very condensed and algebraic details are mostly omitted. *F. J. Dyson.*

Ivanenko, D., and Sokolik, G. The theory of particles of arbitrary isotopic spin and the method of fusion. Dokl. Akad. Nauk SSSR (N.S.) **97**, 635–637 (1954). (Russian)

De Broglie's method of fusion is extended to include isotopic as well as ordinary spin. An equation of Pais

[Physica 19, 869–887 (1953); MR 15, 766] is discussed as a particular case of the theory. *A. J. Coleman.*

Hara, Osamu, and Shimazu, Haruo. A new attempt on the self-energy problem of the photon. Progr. Theoret. Phys. 8, 265–279 (1952).

The idea developed in this paper is that the occurrence in quantum electrodynamics of a (mathematically ambiguous) photon self energy makes it in principle necessary to apply renormalization to the velocity of light. To second order in the electron charge the authors calculate the energy E, velocity c and momentum p of a free photon of given wave vector k and show that the relation $E=pc$ remains valid. It is the correction to c which should be treated as renormalization of the velocity of light. However, this correction cannot be shown to be independent of the wave vector k, so that, unlike for mass and charge renormalization, the term added to the hamiltonian to cancel corrections to c may give rise to new physical effects. Since this term contains probably higher derivatives, the authors propose to quantize the theory anew by a method due to Kanesawa and Koba [same journal 4, 297–311 (1949); MR 11, 765]. *L. Van Hove* (Utrecht).

Rzewuski, Jan. Radiative collisions between two electrons. Acta Phys. Polon. 9, 121–128 (1948).

The author calculates the differential cross-section for the radiative collision of two charged relativistic Dirac particles. Bethe and Heitler's formula for "bremsstrahlung" [Proc. Roy. Soc. London. Ser. A. 146, 83–112 (1934)] is obtained as a special case, when the mass of one of the particles becomes infinitely large. *A. Salam.*

Rzewuski, Jan. The self-energy of scalar mesons in interaction with nuclei. Acta Phys. Polon. 10, 141–150 (1951).

Using Schwinger's formalism [Phys. Rev. (2) 74, 1439–1461 (1948); 75, 651–679 (1949); MR 10, 345, 663], meson self-energy is calculated in the second order, for scalar mesons interacting with the nucleons, with scalar and derivative couplings. The calculation is substantially the same as that carried through by Matthews [ibid. 76, 1657–1674 (1949)] except that it is not realized that the derivative coupling can be completely transformed away [F. J. Dyson, ibid. 73, 929–930 (1948)]. *A. Salam.*

Rzewuski, Jan. Statistical interpretation of the Klein-Gordon equation. Acta Phys. Polon. 11, 1–8 (1951).

It is shown that a consistent quantum-statistical interpretation of the Klein-Gordon equation in its first-quantized form is possible provided Feynman type of boundary conditions [Phys. Rev. (2) 76, 749–759 (1949)] are specified on the hyperplanes $t=\pm\infty$. *A. Salam.*

Rzewuski, Jan. Field theories without divergences. Acta Phys. Polon. 11, 9–24 (1951).

Three methods for constructing divergence-free field theories are discussed. First the author considers the use of an "extended" source, introduced in a Lorentz-invariant manner following McManus and Peierls [McManus, Proc. Roy. Soc. London. Ser. A. 195, 323–336 (1948); MR 10, 664]. An application of Feynman's methods [Phys. Rev. (2) 76, 769–789 (1949); MR 11, 765] makes the calculation of S-matrix elements possible without having to use the conventional second-quantization procedure. It is concluded that the use of the extended source can make only electron self-energy finite, without affecting photon self-energy or vacuum polarization divergences. To remove these, a weight factor is introduced which averages the extended source Lagrangians according to the electron mass. The methods used here are similar to those of Pauli and Villars [Rev. Mod. Phys. 21, 434–444 (1949); MR 11, 301].

The third method investigated for removing divergences is the introduction of Lagrangians with higher-order derivatives of the fields. It is shown that higher-order equations for the electromagnetic (or any other neutral) field are equivalent to the extended-source theory. For charged (electron) fields convergence factors are obtained but the gauge invariance of the theory is lost. It is concluded that all physically interesting possibilities are already exhausted by Lagrangians with first-order derivatives, if the extended-source theory with appropriate weight factors for masses is employed. *A. Salam* (Cambridge, England).

Rzewuski, Jan. On the connection between fields and particles. Acta Phys. Polon. 11, 203–214 (1953).

Consider, for the electromagnetic field, the Lagrangian density, $L=\frac{1}{2}f_{\mu\nu}f_{\mu\nu}+A_\mu\bar{S}_\mu$, where $\bar{S}_\mu$ describes the sources and explicitly equals $\bar{S}_\mu=\sum e_n\int_{-\infty}^{\infty}F(x-\xi^n)\xi_\mu^n d\tau_n$. The invariant function F which describes the extension of the sources differs from zero only in the neighbourhood of ξ^n. Let $\bar{A}_\mu(x)=\int F(x-x')dx'A_\mu(x')$, with analogous definitions for $\bar{f}_{\mu\nu}$. The equations of motion of the source particles are $e_n\sum_m \bar{f}^m{}_{\mu\nu}(\xi^n)\xi_\nu^n=0$, $n=1, 2, \cdots$. These are the Lorentz equations of motion with the inertia term contained in the self-force $e_n\bar{f}^n{}_{\mu\nu}(\xi^n)\xi_\nu^n$ of the nth particle. In this paper the author shows that for processes for which the particles do not come very close to each other, the self-force terms give the electromagnetic mass of the particles. *A. Salam.*

Rzewuski, Jan. Relativistic quantum dynamics of a system of interacting particles. Acta Phys. Polon. 12, 77–80 (1953) = Nuovo Cimento (9) 10, 90–93 (1953).

The two papers are exact transcripts of each other. Consider the action integral

$$-\sum_n k_n\int_{\sigma_1}^{\sigma_2}[1-(v_i^n)^2]^{1/2}dt^n + \tfrac{1}{2}\sum_{n,m} e_n e_m\int_{\sigma_1}^{\sigma_2} v_\mu^n G(q^n-q^m)v_\mu^m dt^n dt^m$$

for a system of interacting particles. Here $v_i=da_i/dt$, $v_0^n=1$ and G is an invariant function. The equations of motion are

$$\frac{d\pi_i^n}{dt}=e_n F^n{}_{i\nu}v_\nu^n,$$

where $\pi_i=kv_i/[1-(v_k^n)^2]^{1/2}$, $F_{i\nu}=\partial A_\nu/\partial q_i-\partial A_i/\partial q_\nu$, and

$$A_\nu(q^n)=\sum_{n\neq m} e_m\int_{\sigma_1}^{\sigma_2}\tfrac{1}{2}[G(q^n-q^m)v_\nu^m+v_\nu^m G(q^n-q^m)]dt^m.$$

If p_k are the free particle limits of π_k for the case of no interaction, the (operator) equations for the π's can be integrated, and in particular a unitary S-matrix constructed, from the relation $p_k^2=S^{-1}p_k^1 S$. Here superscripts 2 and 1 refer to the values of p_k on space-like surfaces σ_2 and σ_1. The methods used are those of Yang and Feldman [Phys. Rev. (2) 79, 972–978 (1950); MR 12, 569]. The author claims that, "the relativistic problem of interacting particles seems to be solved in principle in the framework of quantum dynamics without the use of the field concepts."

A. Salam (Cambridge, England).

Nishiyama, T. On the velocity operator in quantum mechanics. Progr. Theoret. Phys. 7, 417–418 (1952).

For a charged particle represented by a complex quantized field $\phi(x)$, it is proposed to define a velocity operator

$$v(x)=(\hbar/2mi)[\Psi^*(\partial\Psi/\partial x)-(\partial\Psi^*/\partial x)\Psi],$$

where Ψ is defined by the decomposition

$$\phi=\Psi R,\quad \phi^*=R\Psi^*,\quad \Psi\Psi^*=1.$$

The current density operator is then $J=RvR$. The commutation rules and equations of motion for $v(x)$ are exhibited.

F. J. Dyson (Princeton, N. J.).

Rojansky, V. Quantum-mechanical operators. Phys. Rev. (2) 97, 507 (1955).

For a one-dimensional system defined by a coordinate x, let U be the operator $Uf(x)=\int C(s-x)f(s)ds$, with C a fixed function. It is proposed to define the "position operator" X by $X=\frac{1}{2}(xU+Ux)$, the momentum operator P having its usual definition $P=-i\hbar(\partial/\partial x)$. The commutation rule is $XP-PX=i\hbar U$. The idea is that X is the quantity which is actually concerned in a physical measurement of position, and since no position can be measured with perfect accuracy we should require that X have no exact eigenstates.

F. J. Dyson (Princeton, N. J.).

Thermodynamics, Statistical Mechanics

*Jeans, J. H. **The dynamical theory of gases. 4th ed.** Dover Publications, Inc., New York, 1955. ii+444 pp. Paperbound $2.00; clothbound $3.95.

Reprint by photo-offset of the 4th edition [Cambridge, 1925].

Mayer, Joseph E. Two unsolved problems of statistical mechanics. Comm. Pure Appl. Math. 8, 73–83 (1955).

This is an article of expository nature discussing two outstanding problems in modern statistical mechanics. The first one concerns the evaluation of the partition function for (not very dilute) systems of interacting particles; it is stressed that the relevant basic equations are firmly established and that the difficulty is essentially one of mathematical technique. The second problem concerns the approach to statistical equilibrium and the investigation of states of flux; the difficulty is here more of a logical nature, since no general set of equations is known for the description of the phenomenon; the relation of this problem with the question of a general definition of entropy is indicated.

L. Van Hove (Utrecht).

Stueckelberg, E. C. G. Théorème H et unitarité de S. Helv. Phys. Acta 25, 577–580 (1952).

The paper is devoted to the derivation of Boltzmann's H-theorem when the elementary transition probabilities A_{ij} satisfy the conditions $\sum_i A_{ij}=\sum_j A_{ij}=1$ implied by the unitarity of the S-matrix, without verifying the principle of detailed balance $A_{ij}=A_{ji}$. *L. Van Hove* (Utrecht).

Watanabe, Satosi. Symmetry of physical laws. I. Symmetry in space-time and balance theorems. Rev. Mod. Phys. 27, 26–39 (1955).

This paper is concerned with the assumptions needed for the derivation of the H-theorem, and their significance in terms of symmetry properties of physical laws. According to Husimi [Theory of probability and statistics, Kawade Shobo, Tokyo, 1942] and Stückelberg [see the paper reviewed above] these assumptions are $P_{ij}\geqq 0$, $\sum_j P_{ij}=1$, $\sum_i P_{ij}=1$, P_{ij} denoting the transition probabilities. Only the last one, called assumption of inverse renormalization, is not evident. Whereas it is usually derived quantum-mechanically from the unitarity of the S-matrix, it is here shown to be a consequence of the invariance of the equations of motion for time-reversal or for space-and-time inversion. The paper also gives a classification of various tensorial quantities of physical interest according to their inversion properties for space-inversion and time-reversal and studies how suitable averaging may often restore in a limited sense the principle of detailed balance $P_{ij}=P_{ji}$ in cases where it does not hold for the elementary transition probabilities.

L. Van Hove (Utrecht).

Morrey, Charles B., Jr. On the derivation of the equations of hydrodynamics from statistical mechanics. Proc. Nat. Acad. Sci. U. S. A. 40, 317–322 (1954).

After a brief introduction the paper considers the Fourier transforms of the mass, momentum and energy distributions for a large system of interacting particles and shows that every sequence of such systems with increasing numbers of particles and constant density contains a subsequence for which these Fourier transforms converge to finite limits when the variables are expressed in suitable macroscopic units. The multiple distribution functions of groups of particles in position and momentum are then defined and the well known hierarchy of equations which they obey is given. The paper finally gives without proof an explicit solution of the hierarchy of equations, in the form of a series expansion valid at low densities. The author omits to compare it with the well known Ursell-Mayer expansion valid under similar conditions [see, e.g., J. de Boer, Rep. Progr. Phys. 12, 305–374 (1949); MR 12, 308].

L. Van Hove (Utrecht).

Winter, Jacques. Considérations sur la théorie des liquides. Ann. Inst. H. Poincaré 14, 1–33 (1954).

The paper describes in qualitative terms the theoretical ideas of the author on the structure of liquids. These ideas differ from the familiar concept of cell structure, according to which each particle vibrates in the "cell" formed by its nearest neighbors, by the assumption that quantum-mechanical resonances occur between excitations of neighboring particles in their respective cells, and that these resonances extend over fairly large "clusters" of cells and are maintained for fairly long times. Qualitative merits of this concept of "resonating cluster" are indicated, but, except for a crude comparison with the occurrence of long range order in mixed crystals, no serious quantitative analysis is attempted. *L. Van Hove* (Utrecht).

Bayet, Michel, Delcroix, Jean-Loup, et Denisse, Jean-François. Théorie cinétique des plasmas homogènes faiblement ionisés. II. J. Phys. Radium (8) 16, 274–280 (1955).

The study of a perfect Lorentz gas begun in part I [same J. 15, 795–803 (1954); MR 16, 550] is extended to include an evaluation of the coefficient of the square of the electric field strength E in the expansion of the velocity distribution in powers of E. This describes, particularly, the effects of energy exchange between the electric field and the electron gas. *G. Newell* (Providence, R. I.).